CS 3630!

Lecture 9:
A Vacuum Cleaning Robot:
Inference in Factor Graphs

Lecture 8 recap

For our vacuum robot, we looked at

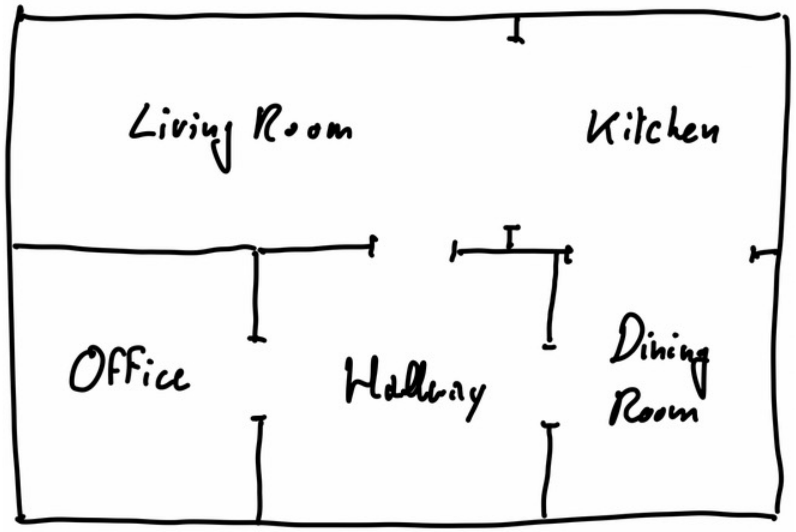
- Simple sensing
- (Dynamic) Bayes Nets
- HMMs as a special case
- Most probable explanation (MPE)
- Factor graphs

Vacuuming robot sensor

- A single sensor that detects light levels, and returns a measurement z:

    - Bright, $z = 2$
    - Medium, $z = 1$
    - Dark, $z = 0$

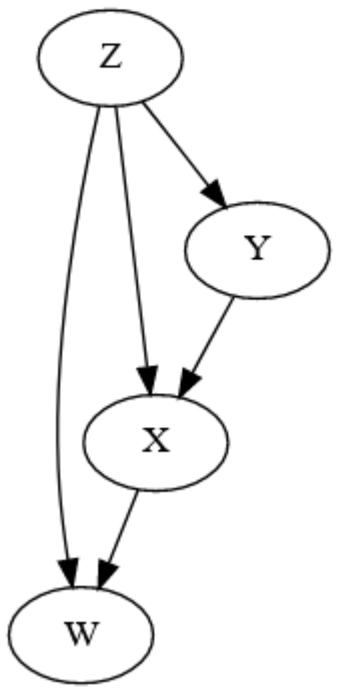
X1	dark	medium	light
Living Room	0.1	0.1	0.8
Kitchen	0.1	0.1	0.8
Office	0.2	0.7	0.1
Hallway	0.8	0.1	0.1
Dining Room	0.1	0.8	0.1



- Sun is to the south, so plenty of light for living room and kitchen.
- Office and Dining room are poorly lit via windows.
- Hallway has no windows and is always dark.

- ***For Hallway, $(z = 0 \,|H) = 0.8$, MLE will do the job!***
- ***For $z = 1, z = 2$, there's really no way to uniquely identify state from one measurement.***

The Magic of Bayes Nets



For a Bayes net with variables $X_1 \ldots X_n$, the joint distribution is given by:

$$\boldsymbol{P(X_1 \ldots X_n) = \prod_i P(X_i | \mathcal{P}(X_i))}$$

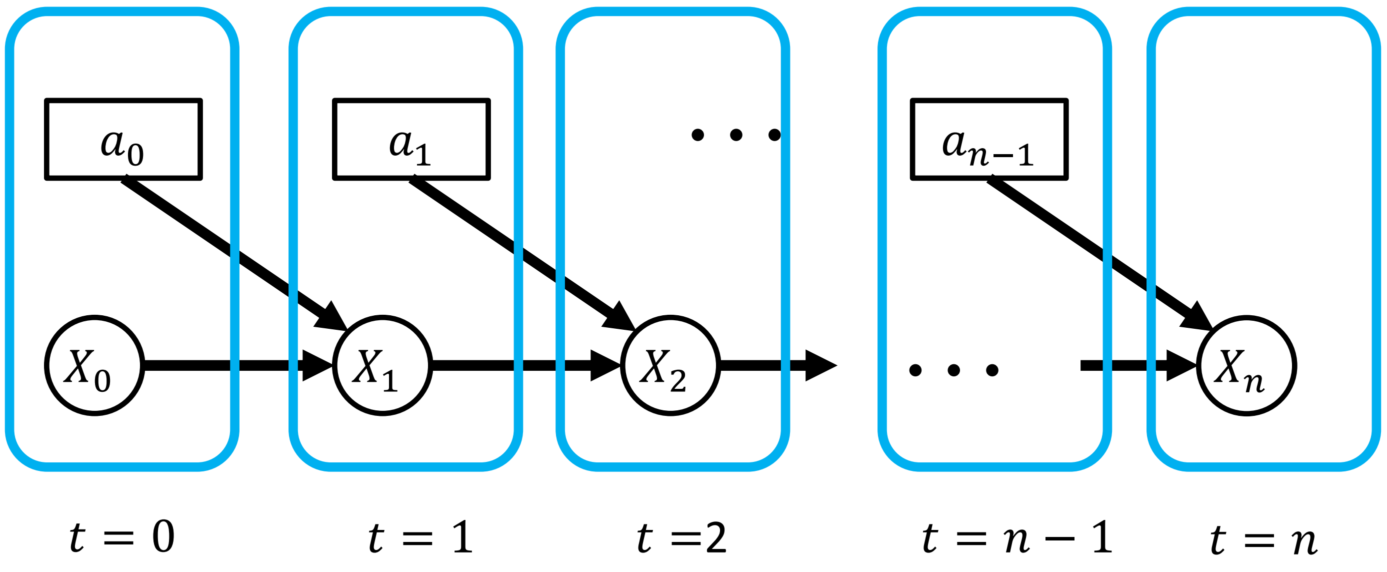
where $\boldsymbol{\mathcal{P}(X_i)}$ denotes the set of parents of node $\boldsymbol{X_i}$

For this specific network, the joint distribution is given by

$$\boldsymbol{P(W, X, Y, Z) = P(W|X, Z) P(X|Y, Z) P(Y|Z) P(Z)}$$

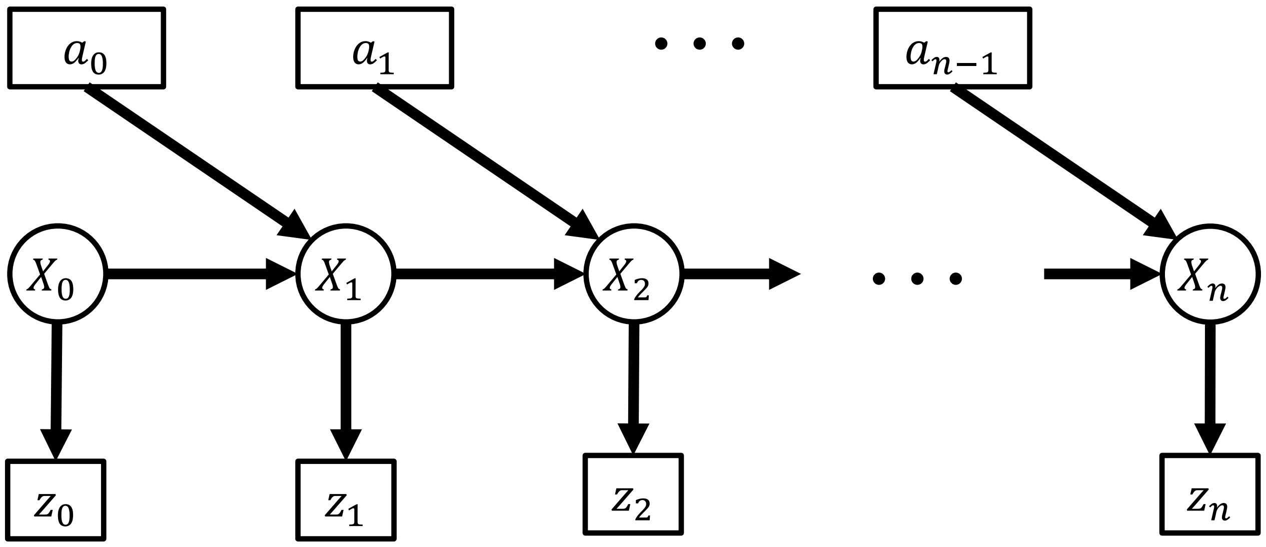
Dynamic Bayes Nets

- Bayes nets can be used to represent systems that evolve over time.
- Our vacuum cleaning robot is an example of such a system, at any time t, we have x_t and a_t and together, these determine (probabilistically) what happens for x_{t+1}.
- A dynamic Bayes net has a simple structure that repeats at each time step:



Hidden Markov Models (HMMs)

- Notice that in the system shown below,
  - we know $\boldsymbol{Z_t = z_t}$ for all $\boldsymbol{t}$
  - We know $\boldsymbol{a_t}$ for all $\boldsymbol{t}$
- We do not know any of $\boldsymbol{X_0 \ldots X_1}$, but we do know that the states form a Markov chain.
- We say that the states, $\boldsymbol{X_0 \ldots X_n}$, are hidden.



HMMs are a good model for speech recognition systems:

- Spoken words behave like a Markov chain (if you know the current word, you know a lot about what will be the next word).
- Measurements are audio signals.

Note: If we increase the relevant history, e.g., so that state X_t depends on $X_{t-1}, X_{t-2} \ldots X_{t-n}$, we have an nth order Markov chain.
Larger n gives better prediction.

Most Probable Explanation

We are given $Z_t = z_t$, and a_t for all t.

For every possible value of $x_0, \dots, x_n$, compute

$$\boldsymbol{P(X, Z, A) = P(Z_0 = z_0 | X_0 = x_0) P(X_0 = x_0) \prod_i P(Z_i = z_i | X_i = x_i) P(X_i = x_i | X_{i-1} = x_{i-1}, a_i)}$$

Our estimate is given by

$$\boldsymbol{X^* = arg\, max_X\, P(X, Z, A)}$$

Not the most efficient algorithm, but in principle, this gets the job done.

Inference in Factor Graphs

We take an HMM, modeled as a Bayes net.

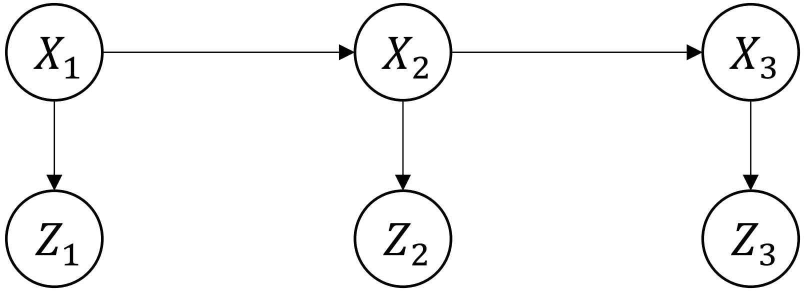
We would like to do MPE as well as full Bayesian inference.

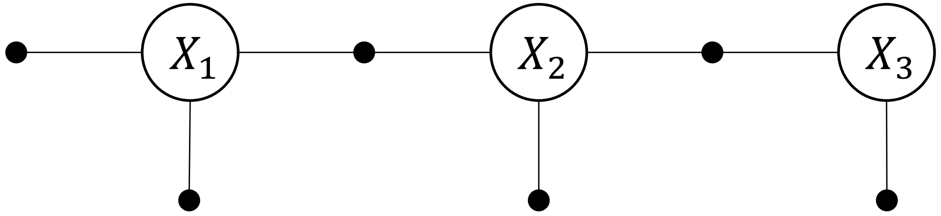
We already know a naïve algorithm for this: brute-force enumeration of the posterior.

When we remove all variables that are given to us we obtain a factor graph.

- ***The factor graph is a great tool to exploit the sparse model structure to obtain computationally efficient inference algorithms.***

Factor Graphs





- Measurements are given – get rid of them!

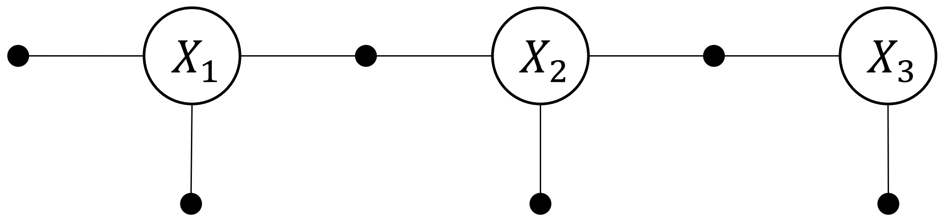
$$P(\mathcal{X}|\mathcal{Z}) \propto P(X_1)L(X_1;z_1)P(X_2|X_1)L(X_2;z_2)P(X_3|X_2)L(X_3;z_3)$$

- This becomes:

$$\phi(\mathcal{X}) = \phi_1(X_1)\phi_2(X_1)\phi_3(X_1,X_2)\phi_4(X_2)\phi_5(X_2,X_3)\phi_6(X_3)$$

Each factor defines a function ϕ which is a function only of its (non-factor node) neighbors.

General definition of Factor graphs



- Bipartite graph of variables and factors

$$\phi(\mathcal{X}) = \prod_i \phi_i(\mathcal{X}_i)$$

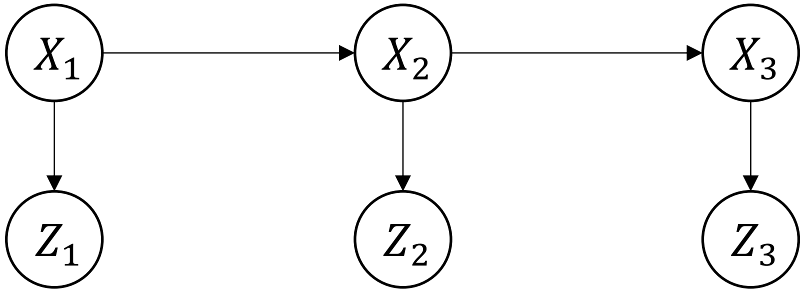
- Each $\mathcal{X}_i$ is the subset of variables connected to factor ϕ_i

Subsets here are:

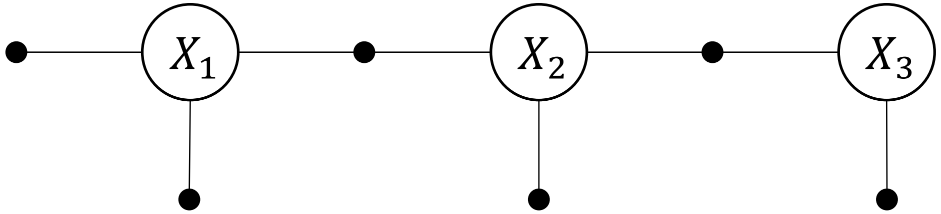
$\mathcal{X}_1 = \{X_1\}$
$\mathcal{X}_2 = \{X_1\}$
$\mathcal{X}_3 = \{X_1, X_2\}$
$\mathcal{X}_4 = \{X_2\}$
$\mathcal{X}_5 = \{X_2, X_3\}$
$\mathcal{X}_6 = \{X_3\}$

Example

$$P(\mathcal{X}|\mathcal{Z}) \propto P(X_1)L(X_1;z_1)P(X_2|X_1)L(X_2;z_2)P(X_3|X_2)L(X_3;z_3)$$



$$\phi(\mathcal{X}) = \phi_1(X_1)\phi_2(X_1)\phi_3(X_1,X_2)\phi_4(X_2)\phi_5(X_2,X_3)\phi_6(X_3)$$

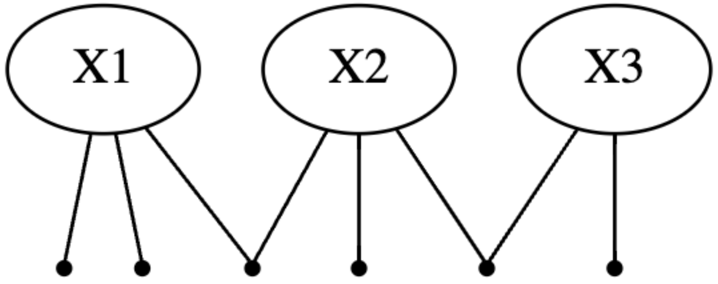


$$\phi_6(X_3) \doteq L(X_3;z_3).$$

Factor Graphs in GTSAM

- See https://www.roboticsbook.org/S34_vacuum_perception.html#factor-graphs-in-gtsam

```
show(graph)
```



$$\phi(X_1, X_2, X_3) = \prod \phi_i(\mathcal{X}_i)$$

Computing with Factor Graphs

- Can evaluate for any X: $\phi(X_1, X_2, X_3) = \prod \phi_i(\mathcal{X}_i)$
- Hence, naïve MPE is also simple, e.g.:

```
mpe_value = 0
mpe_trajectory = None
for x1 in vacuum.rooms:
    for x2 in vacuum.rooms:
        for x3 in vacuum.rooms:
            trajectory = VARIABLES.assignment({X[1]: x1, X[2]: x2, X[3]: x3})
            value = graph(trajectory)
            if value > mpe_value:
                mpe_value = value
                mpe_trajectory = trajectory
print(f"found MPE solution with value {mpe_value:.4f}:")
pretty(mpe_trajectory)
```

```
found MPE solution with value 0.3277:
```

Variable	value
X1	Hallway
X2	Dining Room
X3	Kitchen

Max-Product for MPE

- We can "eliminate" one variable at a time, e.g., X_1:

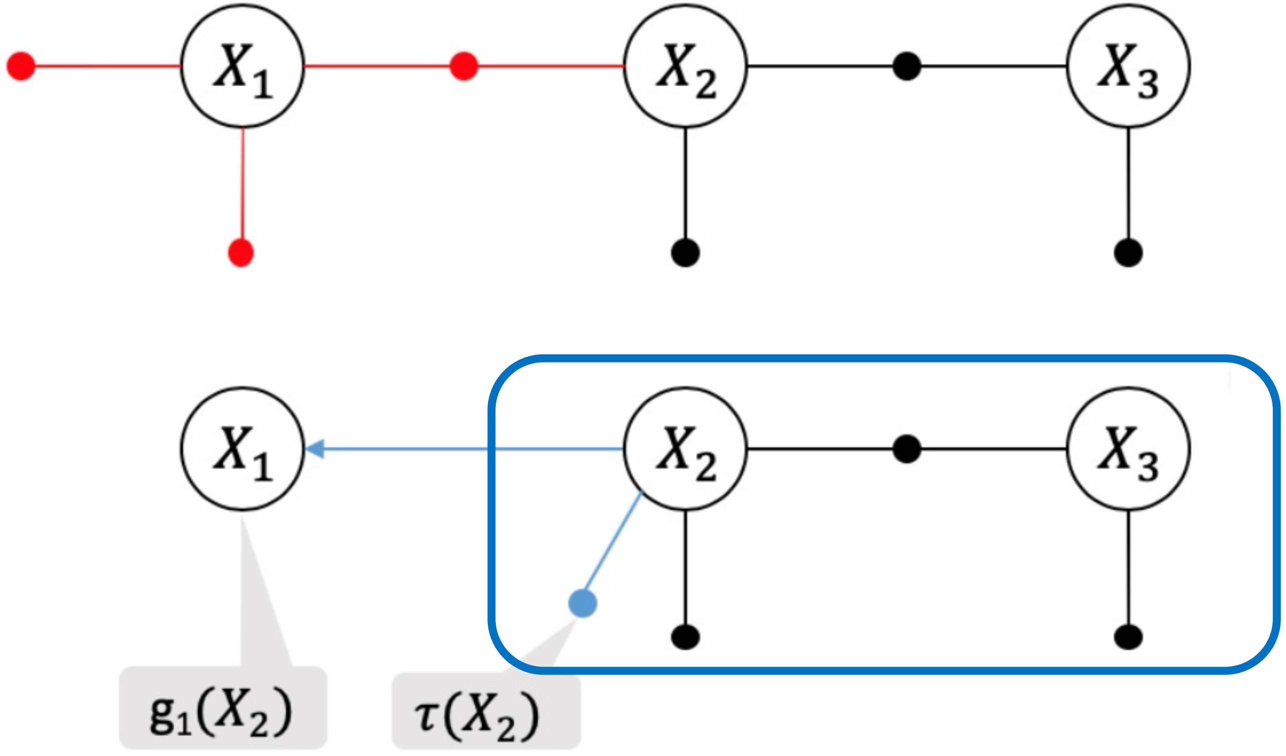
$$\begin{aligned}\max_{\mathcal{X}} \prod \phi_i(\mathcal{X}_i) &= \max_{X_1,X_2,X_3} \phi_1(X_1)\phi_2(X_1)\phi_3(X_1,X_2) \quad \phi_4(X_2)\phi_5(X_2,X_3)\phi_6(X_3) \\ &= \max_{X_2,X_3} \{\max_{X_1} \phi_1(X_1)\phi_2(X_1)\phi_3(X_1,X_2)\} \quad \phi_4(X_2)\phi_5(X_2,X_3)\phi_6(X_3) \\ &= \max_{X_2,X_3} \tau(X_2) \quad \phi_4(X_2)\phi_5(X_2,X_3)\phi_6(X_3)\end{aligned}$$

- The new quantity $\tau(X_2)$ is no longer a function of X_1, as for any given value of X_2 we "memoize" the maximum value of the product of three factors that involve X_1.
- We eliminated X_1 !!!!

A graphical elimination game

- We can represent the elimination of X_1 graphically:

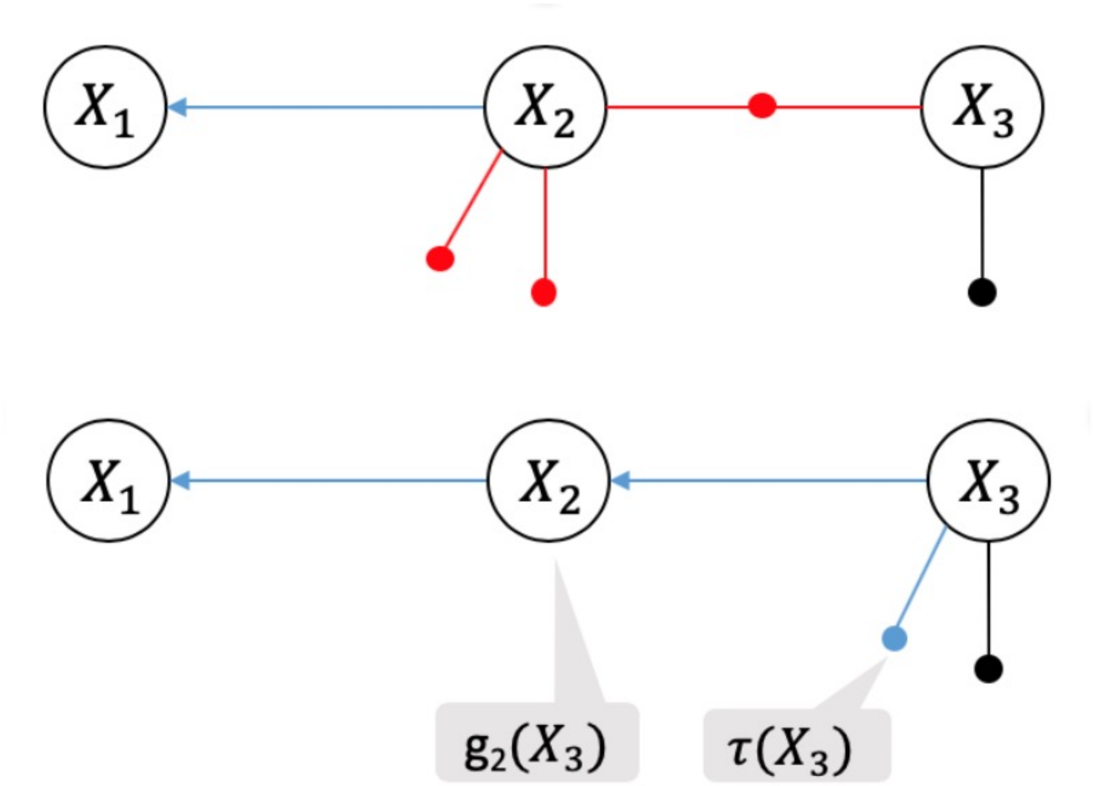
$$\phi(X_1, X_2) = \phi_1(X_1)\phi_2(X_1)\phi_3(X_1, X_2).$$



$$g_1(X_2) = \arg\max_{x_1} \phi(x_1, X_2) \qquad \tau(X_2) = \max_{x_1} \phi(x_1, X_2)$$

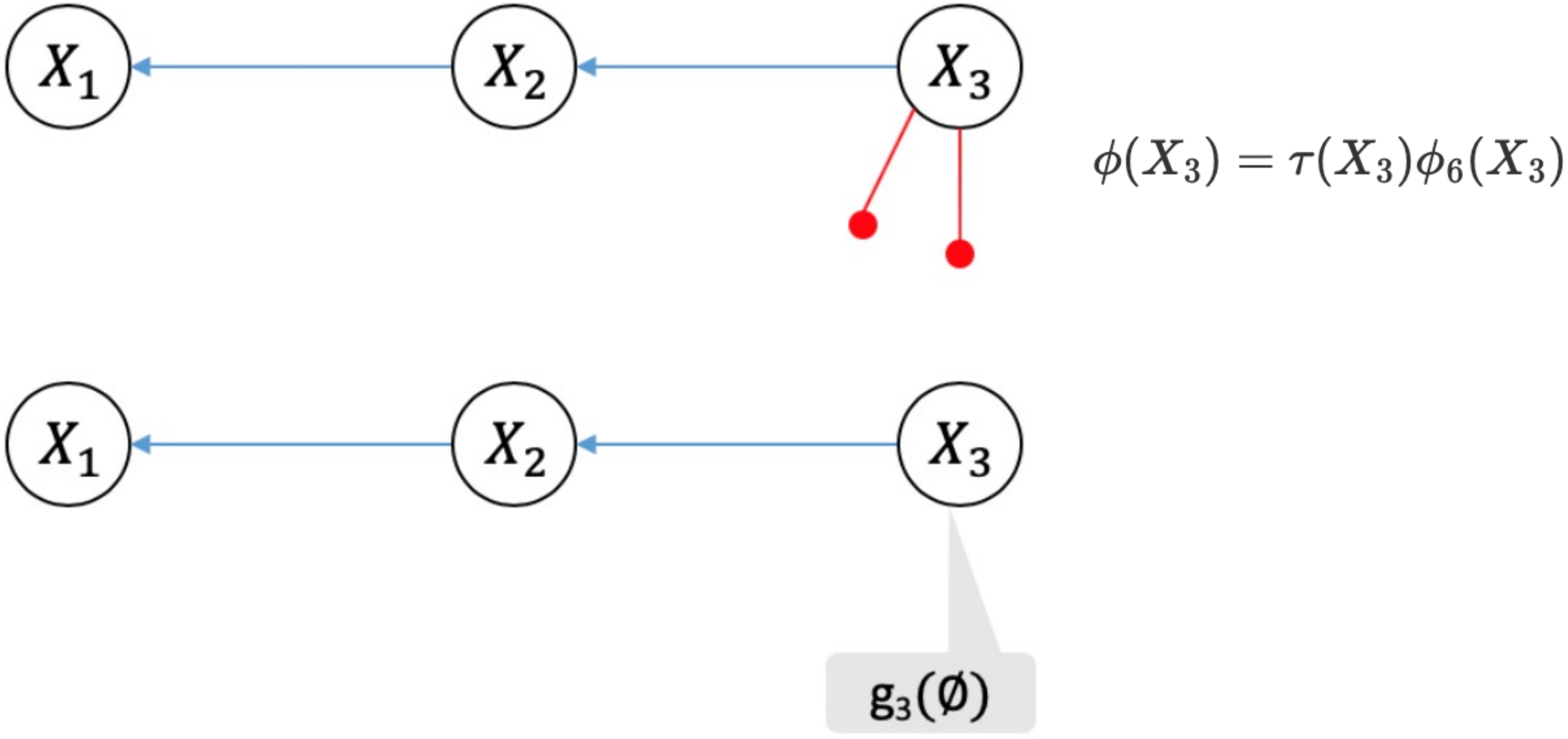
A graphical elimination game

- We then recurse on the new, smaller factor graph by eliminating X_2:



A graphical elimination game

- And finally, we eliminate X_3:



Max-Product Revisited

- We decide on an elimination ordering, then for every elimination step we calculate a product and a maximization:

`MaxProductHMM` $(\Phi_{1:n})$:

- for $j = 1...n$:
  - $g_j(X_{j+1}), \Phi_{j+1:n} \leftarrow \text{CreateLookupTable}(\Phi_{j:n}, X_j)$
  - return $g_1(X_2)g_2(X_3)\ldots g_n(\emptyset)$

`CreateLookupTable` $(\Phi_{j:n}, X_j)$:

- Remove all factors $\phi_i(\mathcal{X}_i)$ that contain X_j
- Form the product factor $\phi(X_j, X_{j+1}) \leftarrow \prod_i \phi_i(\mathcal{X}_i)$
- Eliminate X_j: $g_j(X_{j+1}), \tau(X_{j+1}) \leftarrow \phi(X_j, X_{j+1})$
- Add new factor $\tau(X_{j+1})$ back into the graph $\Phi_{j+1:n}$
- return the lookup table $g_j(X_{j+1})$ and reduced graph $\Phi_{j+1:n}$

Back-substitution

- At the end, we recover the MPE value, but what about the MPE assignment of *X*?
- Lookup tables *g()* to the rescue

MaxProductHMM $(\Phi_{1:n})$:

- for $j = 1...n$:
  - $g_j(X_{j+1}), \Phi_{j+1:n} \leftarrow \mathrm{CreateLookupTable}(\Phi_{j:n}, X_j)$
- return $g_1(X_1)g(X_2)\ldots g_n(\emptyset)$

Max-product for MP in GTSAM

- While I expect you to understand the max-product algorithm, the implementation of is conveniently done by us in GTSAM:

```
mpe = graph.optimize()
pretty(mpe)
```

Variable	value
X1	Hallway
X2	Dining Room
X3	Kitchen

Sum-product for HMMs

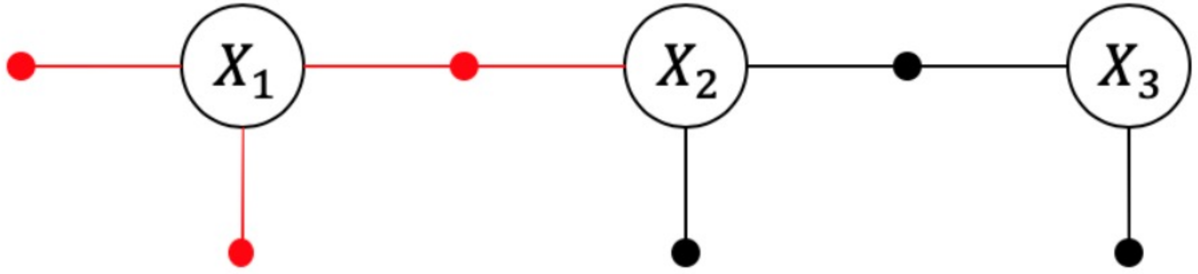
- Similar dynamic programming idea:

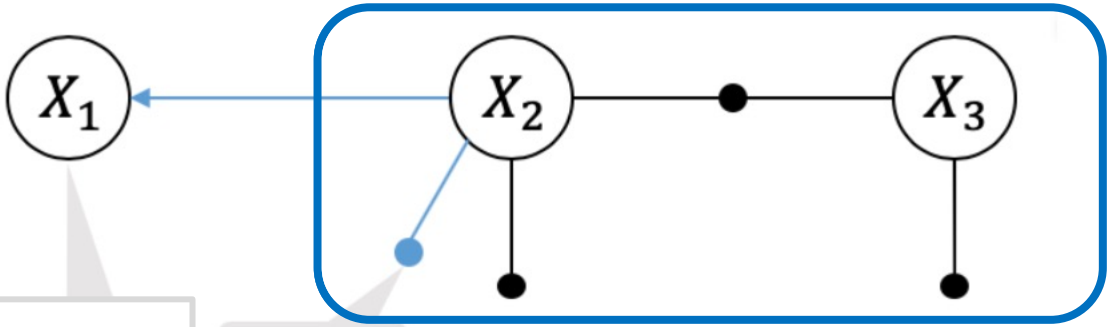
$$
\begin{aligned}
P(\mathcal{X}|\mathcal{Z}) &\propto \prod \phi_i(\mathcal{X}_i) \\
&\propto \phi_1(X_1)\phi_2(X_1)\phi_3(X_1, X_2)\phi_4(X_2)\phi_5(X_2, X_3)\phi_6(X_3) \\
&\propto \{\phi_1(X_1)\phi_2(X_1)\phi_3(X_1, X_2)\}\ \ \phi_4(X_2)\phi_5(X_2, X_3)\phi_6(X_3) \\
&\propto \{P(X_1|X_2, \mathcal{Z})\tau(X_2)\}\ \ \phi_4(X_2)\phi_5(X_2, X_3)\phi_6(X_3) \\
&= P(X_1|X_2, \mathcal{Z})\ \ P(X_2, X_3|\mathcal{Z})
\end{aligned}
$$

Sum-product looks the same graphically:

- But eliminating now involves a sum and a conditional

$$\phi(X_1, X_2) = \phi_1(X_1)\phi_2(X_1)\phi_3(X_1, X_2).$$





$$P(X_1|X_2) = \frac{\phi_1(X_1)\phi_2(X_1)\phi_3(X_1, X_2)}{\tau(X_2)}.$$

$\tau(X_2)$

$$\tau(X_2) \doteq \sum_{X_1} \phi_1(X_1)\phi_2(X_1)\phi_3(X_1, X_2)$$

Comparing max and sum-product:

- Almost identical, replace max with sum:

MaxProductHMM ($\Phi_{1:n}$):

- for $j = 1 \ldots n$:
    - $g_j(X_{j+1}), \Phi_{j+1:n} \leftarrow \text{CreateLookupTable}(\Phi_{j:n}, X_j)$
    - return $g_1(X_2)g_2(X_3)\ldots g_n(\emptyset)$

CreateLookupTable ($\Phi_{j:n}, X_j$):

- Remove all factors $\phi_i(\mathcal{X}_i)$ that contain X_j
- Form the product factor $\phi(X_j, X_{j+1}) \leftarrow \prod_i \phi_i(\mathcal{X}_i)$
- Eliminate X_j: $g_j(X_{j+1}), \tau(X_{j+1}) \leftarrow \phi(X_j, X_{j+1})$
- Add new factor $\tau(X_{j+1})$ back into the graph $\Phi_{j+1:n}$
- return the lookup table $g_j(X_{j+1})$ and reduced graph $\Phi_{j+1:n}$

SumProductHMM ($\Phi_{1:n}$):

- for $j = 1 \ldots n$:
    - $P(X_j|X_{j+1}), \Phi_{j+1:n} \leftarrow \text{ApplyChainRule}(\Phi_{j:n}, X_j)$
    - return Bayes net $P(X_1|X_2)P(X_2|X_3)\ldots P(X_n)$

ApplyChainRule ($\Phi_{j:n}, X_j$):

- Remove all factors $\phi_i(\mathcal{X}_i)$ that contain X_j
- Create product factor $\phi(X_j, X_{j+1}) \leftarrow \prod_i \phi_i(\mathcal{X}_i)$
- Factorize the product $P(X_j|X_{j+1})\tau(X_{j+1}) \leftarrow \phi(X_j, X_{j+1})$
- Add the new factor $\tau(X_{j+1})$ back into the graph $\Phi_{j+1:n}$
- return the conditional $P(X_j|X_{j+1})$ and reduced graph $\Phi_{j+1:n}$

- Spot the differences ☺

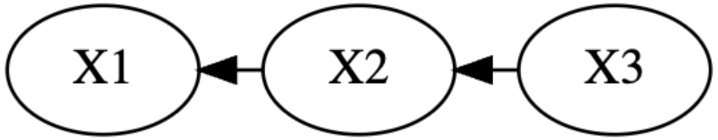
Max and sum-product in GTSAM

- ***optimize*** yields an assignment, ***sumProduct*** yields a distribution!

```
mpe = graph.optimize()
pretty(mpe)
```

Variable	value
X1	Hallway
X2	Dining Room
X3	Kitchen

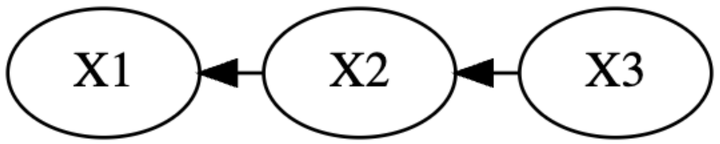
```
posterior = graph.sumProduct()
show(posterior, hints={"X": 1})
```



Power of posteriors

- *We can sample from the posterior:*

```
posterior = graph.sumProduct()
show(posterior, hints={"X": 1})
```




```
sample = posterior.sample()
pretty(sample)
```

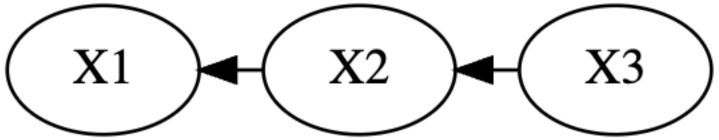
✓ 0.1s

Variable	value
X1	Hallway
X2	Dining Room
X3	Kitchen

Power of posteriors

- *We can sample from the posterior, e.g., sample 1000 alternate histories:*

```
posterior = graph.sumProduct()
show(posterior, hints={"X": 1})
```




```
sample = posterior.sample()
pretty(sample)
```

✓ 0.1s

Variable	value
X1	Hallway
X2	Dining Room
X3	Kitchen

```
counts = np.zeros((3, 5))
num_samples = 1000
for i in range(num_samples):
    sample = posterior.sample()
    for k in range(1,3+1):
        room_index = sample[key]
        counts[k-1][room_index] += 1 # base 0!
```

(variable) room_index: Any

✓ 0.9s

```
pd.DataFrame(data=100*counts/num_samples,
             index=range(1, N+1), columns=vacuum.rooms)
```

✓ 0.7s

	Living Room	Kitchen	Office	Hallway	Dining Room
1	2.0	2.3	2.6	82.5	10.6
2	0.5	3.8	0.6	4.5	90.6
3	5.0	91.9	0.6	0.0	2.5

Next Lecture

- Use the power of this full posterior inference (sum product!) together with a reward/cost framework to act optimally.