CS 3630

L21: Car Kinematics

Self-Driving Cars

The Kinematics of Cars

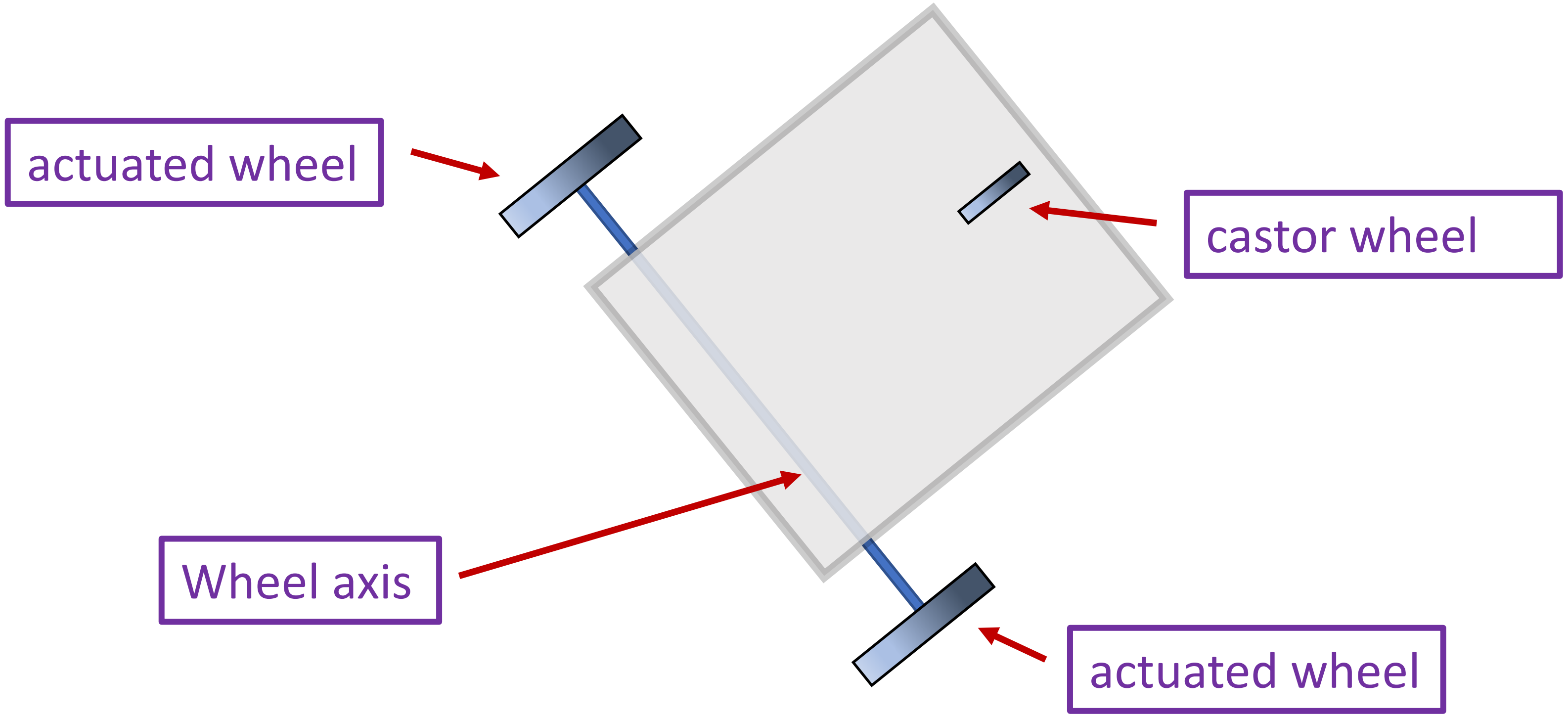
Cars are slightly more complicated than differential drive robots:

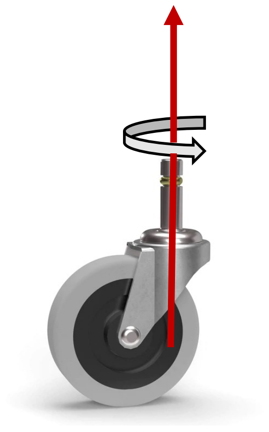
- Cars cannot “turn in place” – they have a finite turning radius.
- Most cars have front-wheel steering, and rear wheels with a fixed orientation – this will require a bit of (not too complicated) geometry.
- Inputs to a car are:
  - ***Steering direction***, which is determined by the angle of a steering wheel for the driver,
  - ***Forward speed***, which is determined by accelerator.
- For real cars, the input is more accurately modeled in terms of acceleration, rather than speed, but for most purposes, treating the car as a velocity-controlled vehicle (and thereby ignoring physics) is a useful abstraction.

Differential Drive Robots

Differential drive robots (aka DDRs):

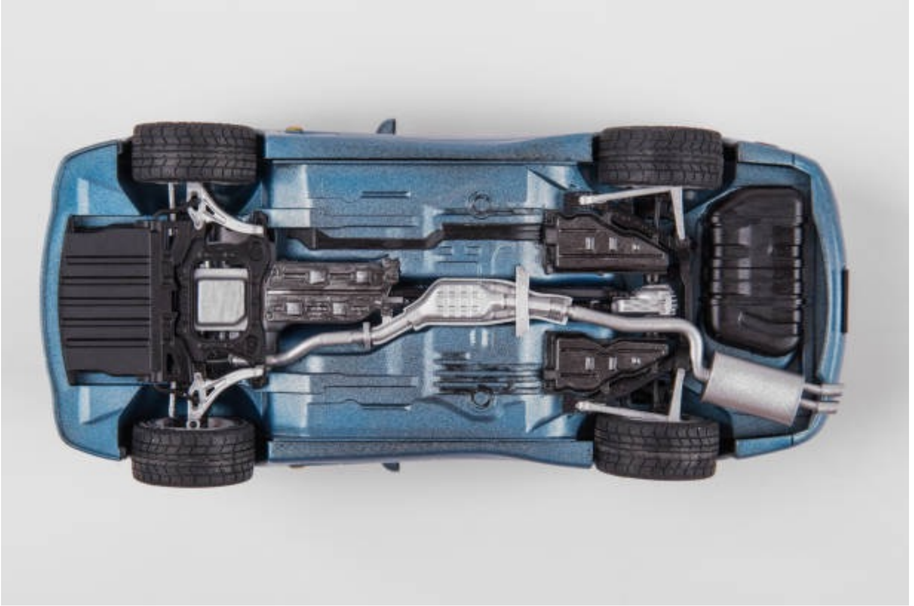
- Two actuated wheels that share an axis
- A castor wheel that rotates freely, mainly to stabilize the robot (three points define a plane – castor wheel keeps the robot from tipping over).





A castor wheel is able to spin freely about the vertical axis.

Cars



Cars have four wheels, and all four matter.

- All four wheels (should) roll without slipping.
- Only the front wheels can turn.
- No castor wheels: rear wheels have a fixed orientation.
- Finite turning radius: cannot spin in place.
- We can develop a simpler model for car kinematics that captures the essential geometry.

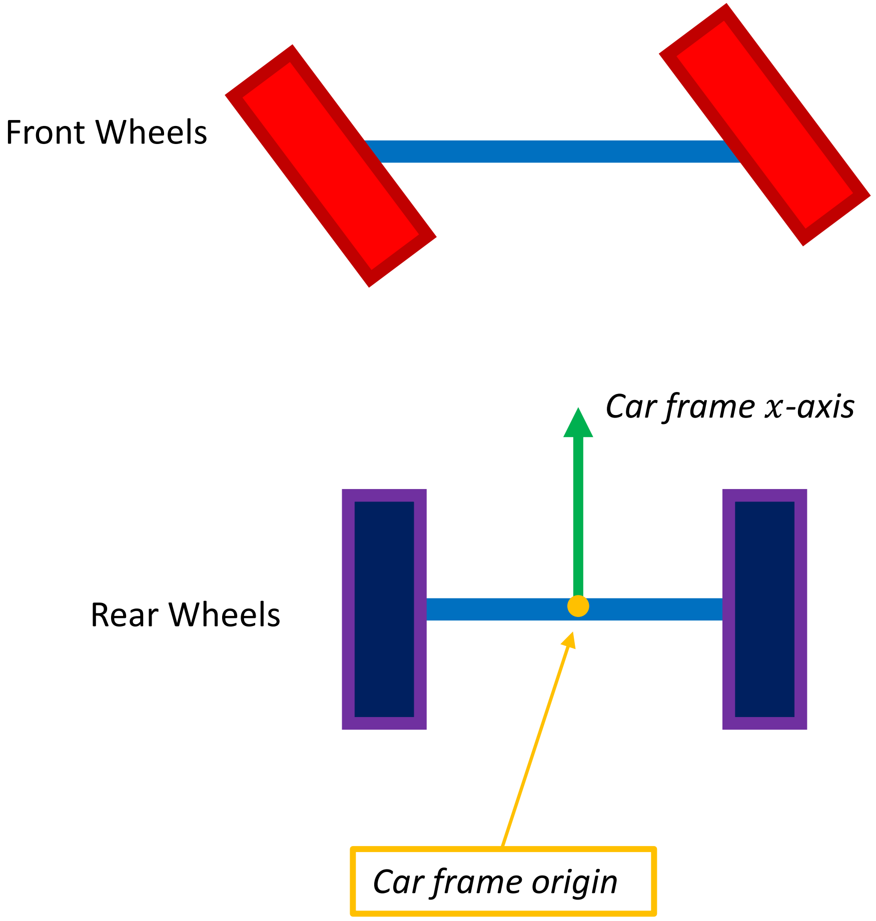
Modeling Front-Wheel Steering

- Typically, cars have front-wheel steering, meaning that turning the steering wheel changes the direction of the two front wheels.

- We assign a coordinate frame to the car:
  - Origin is at the midpoint along the rear axle.
  - Direction of x-axis is perpendicular to the rear axle.

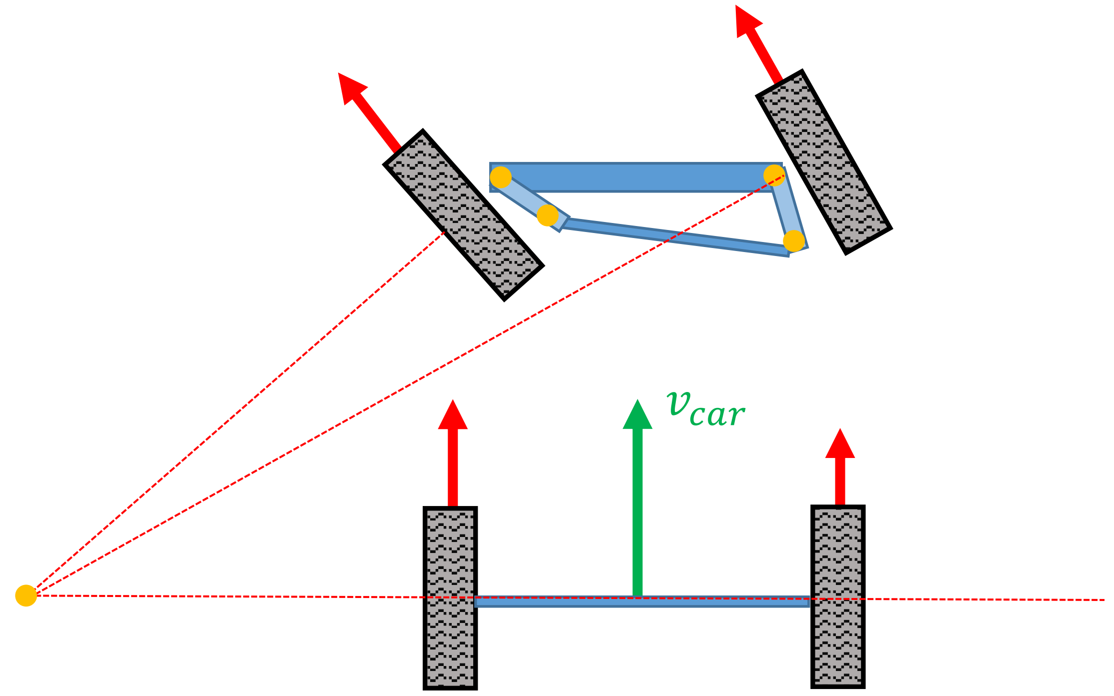
- The linear velocity of the car is defined to be the velocity of the origin of the car frame. This velocity is <u>*always*</u> in the direction of the car frame x-axis.

- This linear speed is denoted by v_{car}.



Ackermann Steering

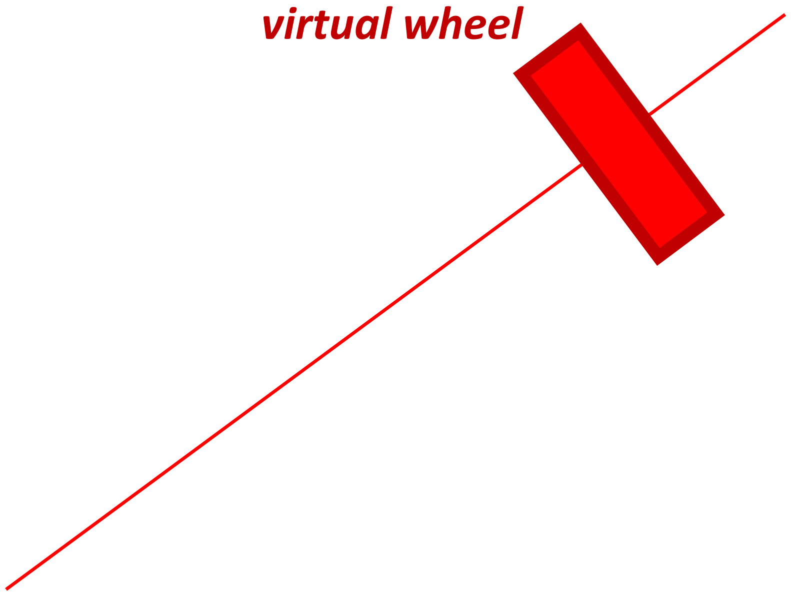
- When a car makes a turning maneuver, in order to avoid slipping (or skidding), the inner and outer wheels must (instantaneously) trace out circles of different radii, but with the same center.
- The instantaneous velocity of each wheel is tangent to the circle that it traces.

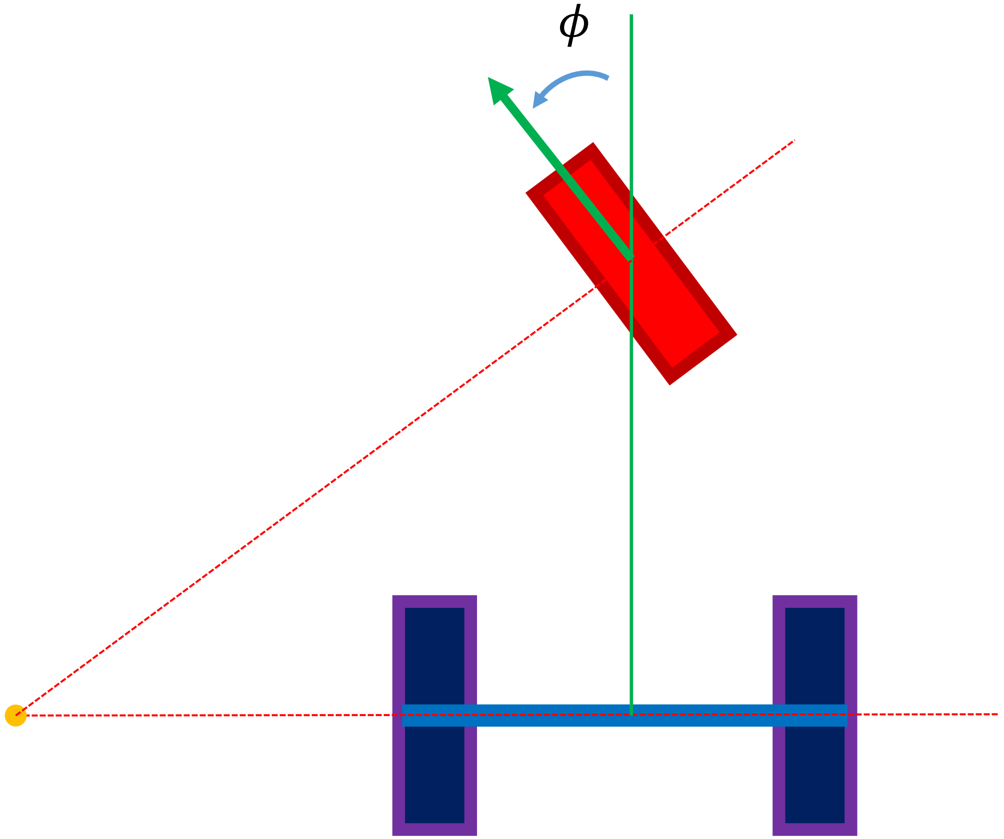


The center point of the rear axle moves with the same velocity as the two rear wheels.

Ackermann Steering (modeling)

- To simplify the geometry, we typically consider a virtual wheel, placed at the midpoint between the two front wheels.
- We'll denote by ϕ the angle of the virtual wheel w.r.t. the car's x-axis.



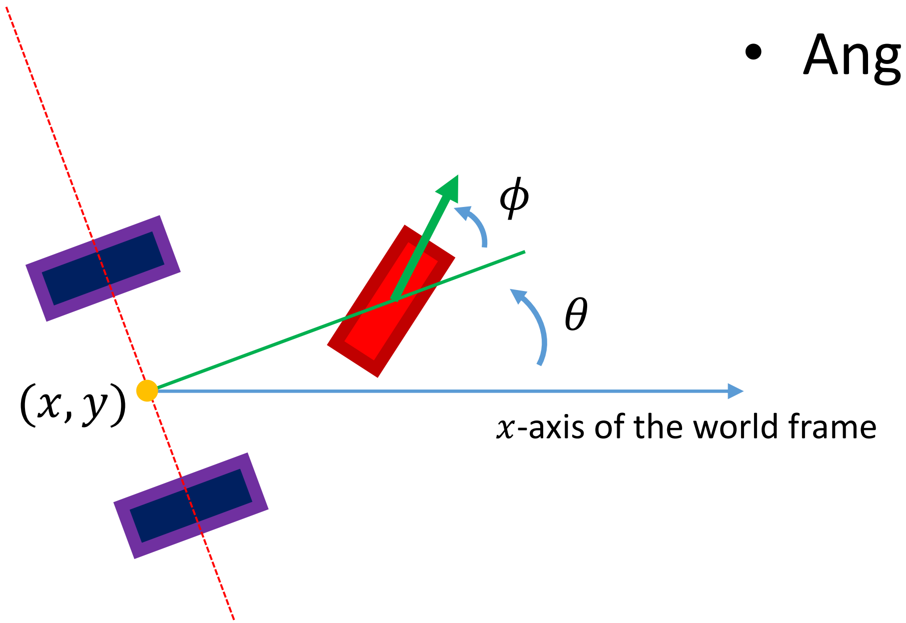


Differential Kinematics

- The configuration of the car can be represented by $q = (x, y, \theta, \phi)$, where
  - x, y defines the origin of the car frame (midpoint of rear axle).
  - θ is the angle from the x-axis of the world frame to the x-axis of the car frame.
  - ϕ is the steering angle.

We'll model the car as having two inputs:

- Linear speed in the direction of the car x-axis: $u_1 = v_{car}$
- Angular turning rate of the virtual wheel: $\mathrm{u}_2 = \dot{\phi}$



The car's differential kinematics are specified by the equation:

$$\dot{q} = f(q, u_1, u_2)$$

Differential Kinematics

Most of this is easy.

- For the linear velocity of the car, expressed w.r.t. the world coordinate frame:

$$\dot{x} = v_{car} \cos\theta = u_1 \cos\theta$$
$$\dot{y} = v_{car} \sin\theta = u_1 \sin\theta$$

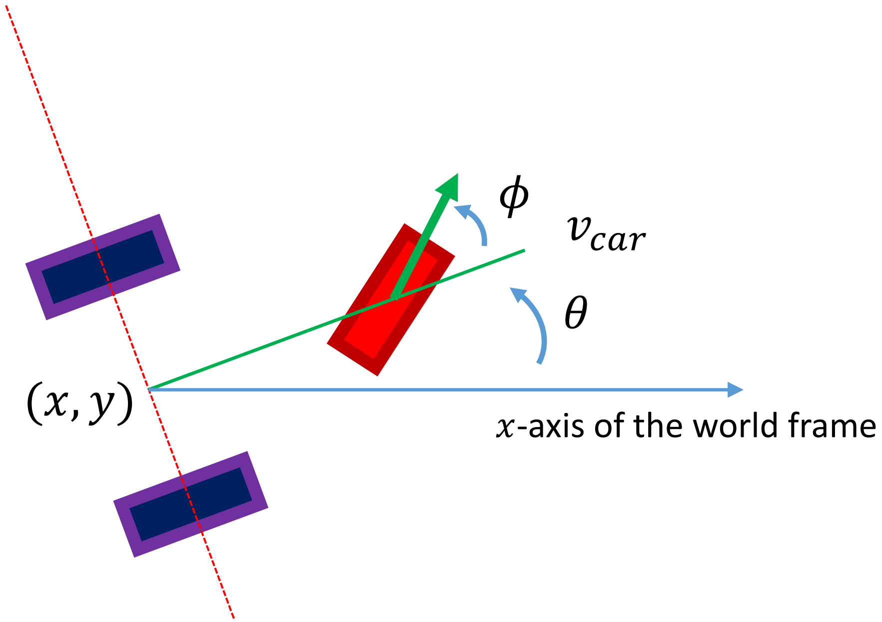
- For the steering angle:

$$\mathrm{u}_2 = \dot{\phi}$$

Note that we have direct control over the rate of change of the steering angle, but only indirect control of the car's forward velocity (since it depends on θ).

NOTE:

- ***With this model, we cannot instantaneously choose the steering angle ϕ.***
- ***We can choose $\dot{\phi}$, and by integrating over time, we can achieve a desired steering angle ϕ.***



Differential Kinematics

For $\dot{\theta}$, first solve for the velocity $v_{tangent}$ perpendicular to the car direction of motion:

$$v_{wheel} \cos\phi = v_{car} \rightarrow v_{wheel} = \frac{v_{car}}{\cos\phi}$$

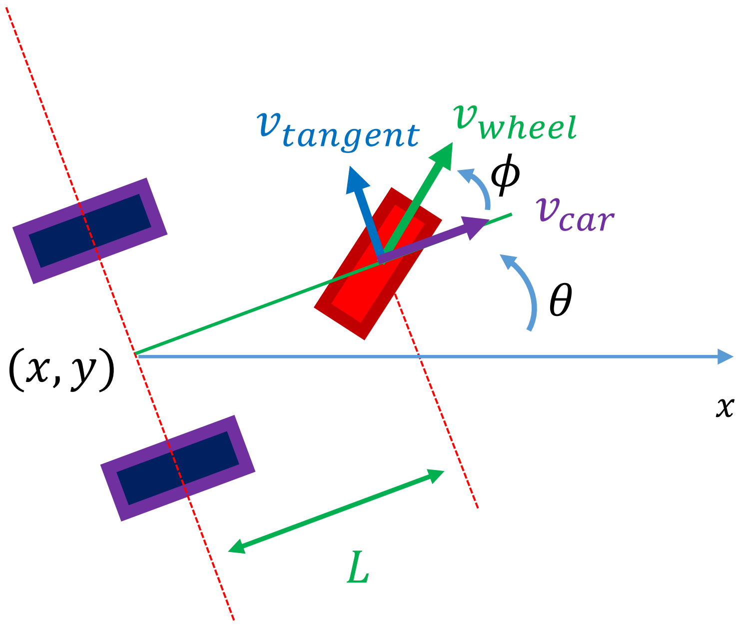
$$v_{wheel} \sin\phi = v_{tangent}$$

Combining these to eliminate v_{wheel} we obtain:

$$\frac{v_{car}}{\cos\phi} \sin\phi = v_{car} \tan\phi = v_{tangent}$$

Now use $L\dot{\theta} = v_{tangent}$ and we arrive to

$$\dot{\theta} = \mathrm{v}_{\mathrm{car}} \frac{1}{L} \tan\phi = u_1 \frac{1}{L} \tan\phi$$



Denote by L the baseline distance, from rear axle to center of steering wheel.

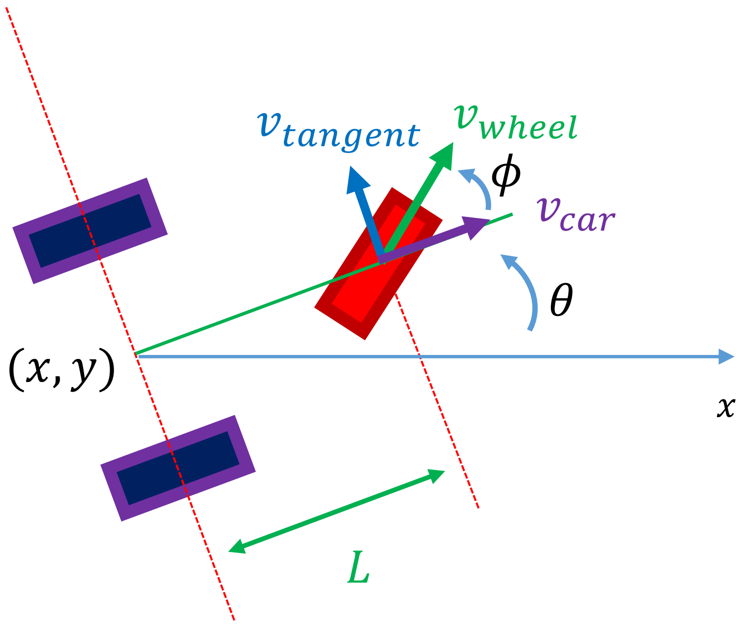
NOTE:

- ***We cannot instantaneously choose θ.***

Differential Kinematics

We can write the differential kinematics as:

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{L}\tan\phi \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$



In which

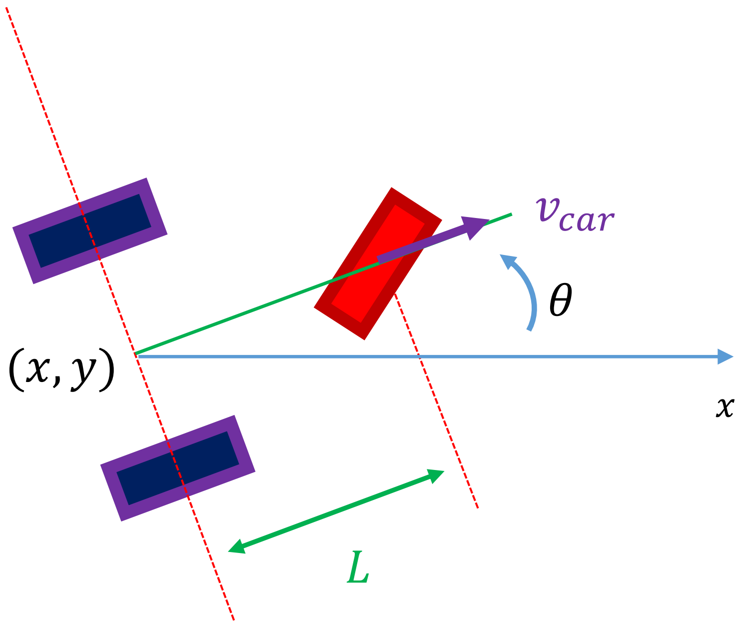
- $q = (x, y, \theta, \phi)$
- $u_1 = v_{car}$
- $u_2 = \dot{\phi}$

Differential Kinematics: simplified form

- It is convenient to write the differential kinematics considering the car angular velocity as an input.
- Instead of controlling θ indirectly via

$$\dot{\theta} = u_1 \frac{1}{L} \tan\phi, \qquad \dot{\phi} = u_2$$

we assume that we can directly control ϕ (instead of $\dot{\phi}$) and define $u_2 = \dot{\theta}$ (it's easy to solve for the ϕ that achieves this).



- We now write the differential kinematics as:

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

In which

- $q = (x, y, \theta)$
- $u_1 = v_{car}$
- $u_2 = \dot{\theta}$, the angular velocity of car frame

This is reasonable if we can turn the sterring wheel quickly, relative to the forward speed of the car.