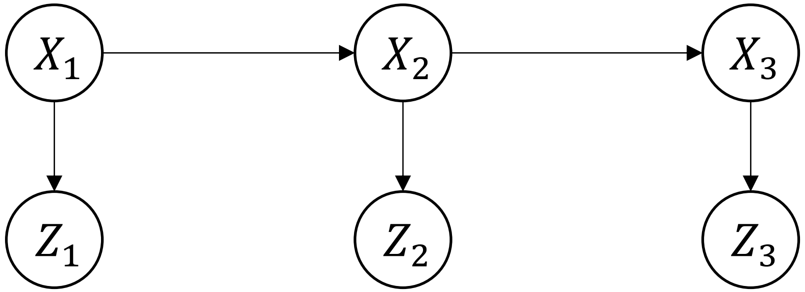
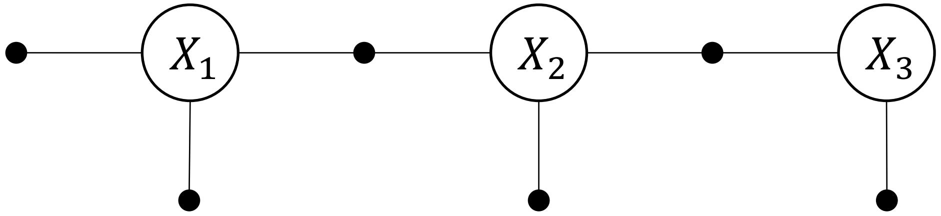
CS 3630!

Lecture 10: Markov Decision Processes

Lecture 9 Recap

Factor Graphs





- Measurements are given – get rid of them!

$$P(\mathcal{X}|\mathcal{Z}) \propto P(X_1)L(X_1; z_1)P(X_2|X_1)L(X_2; z_2)P(X_3|X_2)L(X_3; z_3)$$

- This becomes:

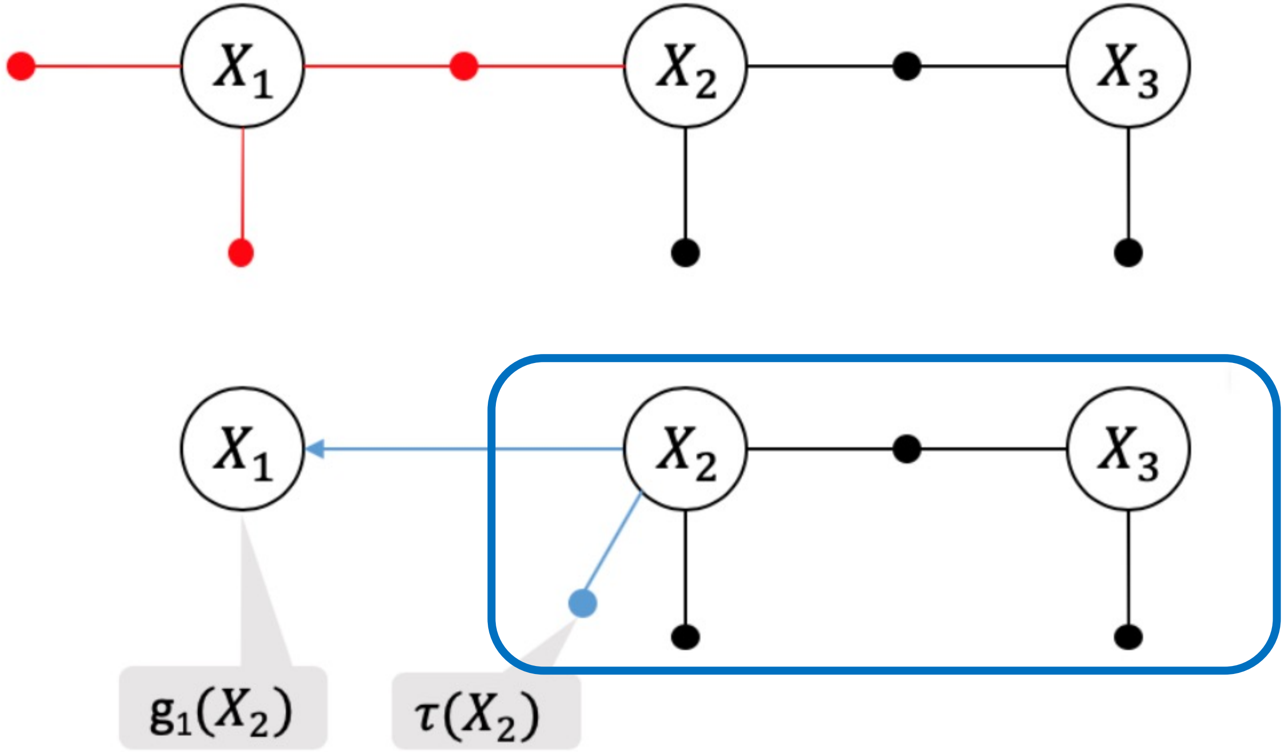
$$\phi(\mathcal{X}) = \phi_1(X_1)\phi_2(X_1)\phi_3(X_1, X_2)\phi_4(X_2)\phi_5(X_2, X_3)\phi_6(X_3)$$

Each factor defines a function ϕ which is a function only of its (non-factor node) neighbors.

MPE via max-product

- Eliminate one variable at a time by forming product, then max:

$$\phi(X_1, X_2) = \phi_1(X_1)\phi_2(X_1)\phi_3(X_1, X_2).$$

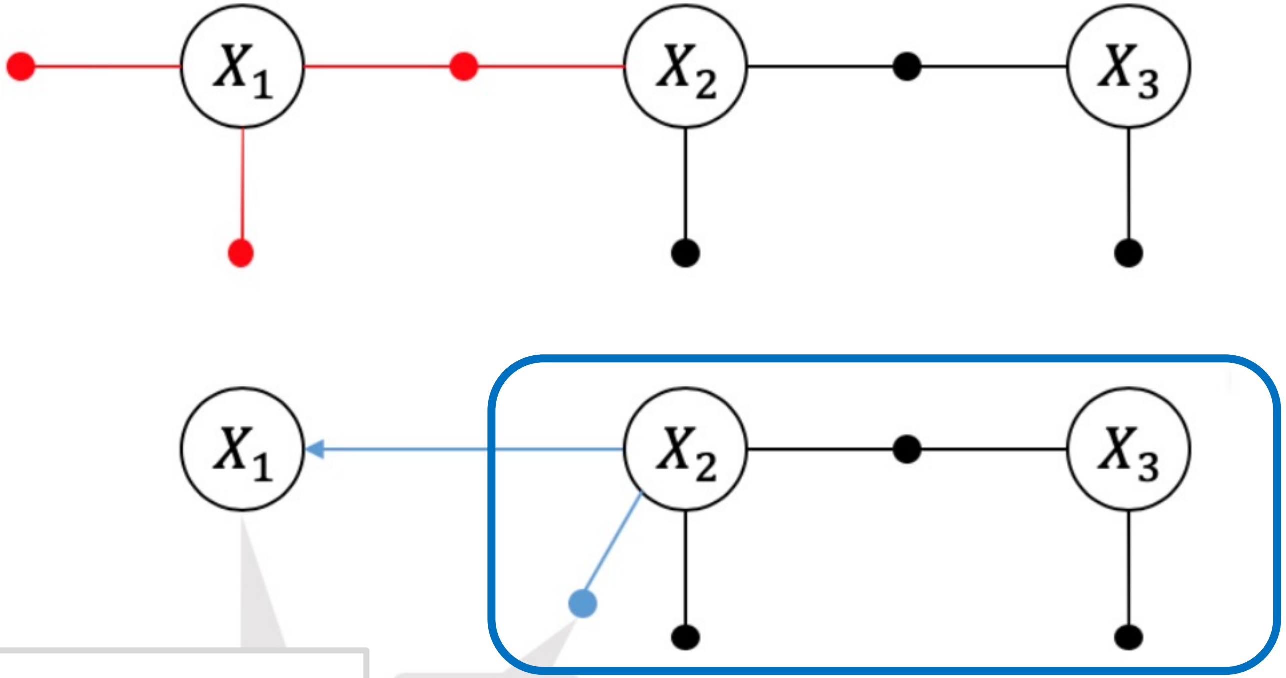


$$g_1(X_2) = \arg\max_{x_1} \phi(x_1, X_2) \qquad \tau(X_2) = \max_{x_1} \phi(x_1, X_2)$$

Posterior via sum-product:

- Eliminate one variable at a time by forming product, then sum:

$$\phi(X_1, X_2) = \phi_1(X_1)\phi_2(X_1)\phi_3(X_1, X_2).$$



$$P(X_1|X_2) = \frac{\phi_1(X_1)\phi_2(X_1)\phi_3(X_1, X_2)}{\tau(X_2)}.$$

$\tau(X_2)$

$$\tau(X_2) \doteq \sum_{X_1} \phi_1(X_1)\phi_2(X_1)\phi_3(X_1, X_2)$$

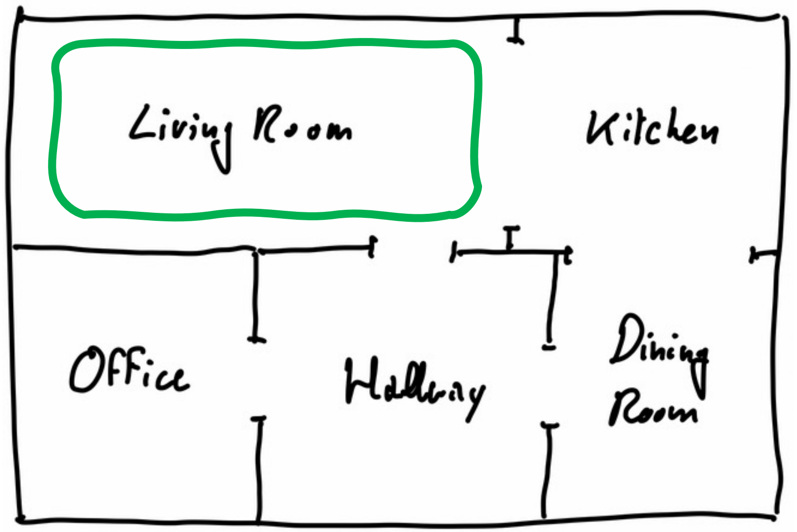
Markov Decision Processes

- Planning is the process of choosing which actions to perform.
- In order to plan effectively, we need quantitative criteria to evaluate actions and their effects.
- MDPs include a reward function that characterizes the immediate benefit of applying an action.
- Policies describe how to act in a given state.
- The value function characterizes the long-term benefits of a policy.
- We assume that the robot is able to ***know*** its current state with certainty.

➢ ***We will see how to define reward functions and use these to compute optimal policies for MDPs.***

Reward Functions

- Most general form depends on current state, action, and next state:
$$R: \mathcal{X} \times \mathcal{A} \times \mathcal{X} \to \mathbb{R}$$
- In our example, we just care about where we end up after taking an action:




```
def reward_function(state:int, action:int, next_state:int):
    """Reward that returns 10 upon entering the living room."""
    return 10.0 if next_state == "Living Room" else 0.0

print(reward_function("Kitchen", "L", "Living Room"))
print(reward_function("Kitchen", "L", "Kitchen"))
```

```
10.0
0.0
```

Expected Reward

- A greedy way to act would be to calculate the immediate expected reward for every possible action:

$$\bar{\boldsymbol{R}}(\boldsymbol{x}, \boldsymbol{a}) = \boldsymbol{E}[\boldsymbol{R}(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{X}')]$$

- Since we know the transition probabilities, we can easily compute this:

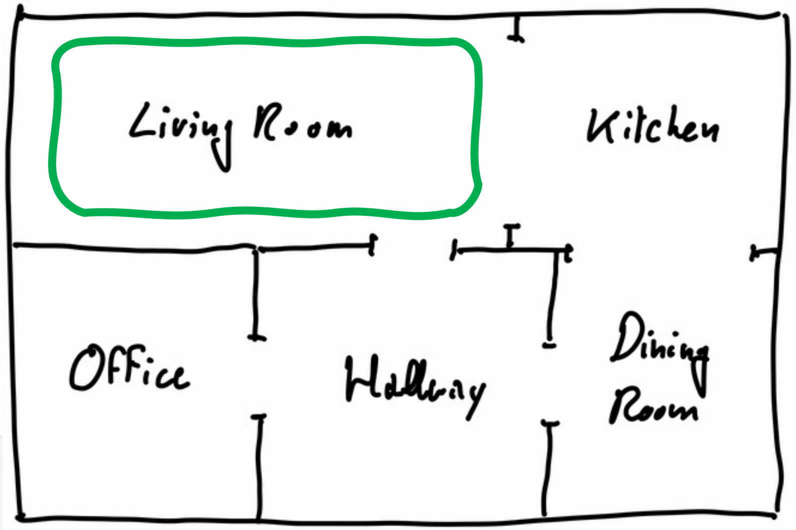
$$\bar{R}(x, a) \doteq E[R(x, a, X')] = \sum_{x'} P(x'|x, a) R(x, a, x')$$

- We then have a simple **greedy planning** algorithm:

$$a^* = \arg\max_{a \in \mathcal{A}} E[R(X_t, a, X_{t+t})]$$

Example

- The expected immediate reward for all four actions in the Kitchen:




```
x = vacuum.rooms.index("Kitchen")
for a in range(4):
    print(f"Expected reward ({vacuum.rooms[x]}, {vacuum.action_space[a]}) = {T[x,a] @ R[x,a]}")
```

✓ 0.9s

```
Expected reward (Kitchen, L) = 8.0
Expected reward (Kitchen, R) = 0.0
Expected reward (Kitchen, U) = 0.0
Expected reward (Kitchen, D) = 0.0
```

- Hence, when in the kitchen, always do L !
- This is a greedy planning algorithm

Utility

$U: \mathcal{A}^n \times \mathcal{X}^{n+1} \to \mathbb{R}$

$$U(a_1, \ldots, a_n, x_1, \ldots x_{n+1}) = R(x_1, a_1, x_2) + \gamma R(x_2, a_2, x_3) + \cdots \gamma^{n-1} R(x_n, a_n, x_{n+1})$$

- Because actions are uncertain, let's look further into the future!
- Introduce a discount factor γ to
  - still bias towards more immediate payoff;
  - allow infinite time horizons:

$$U(a_1, \ldots, a_n, x_1, \ldots x_{n+1}) = \sum_{i=1}^{\infty} \gamma^{i-1} R(x_i, a_i, x_{i+1})$$

Expected Utility

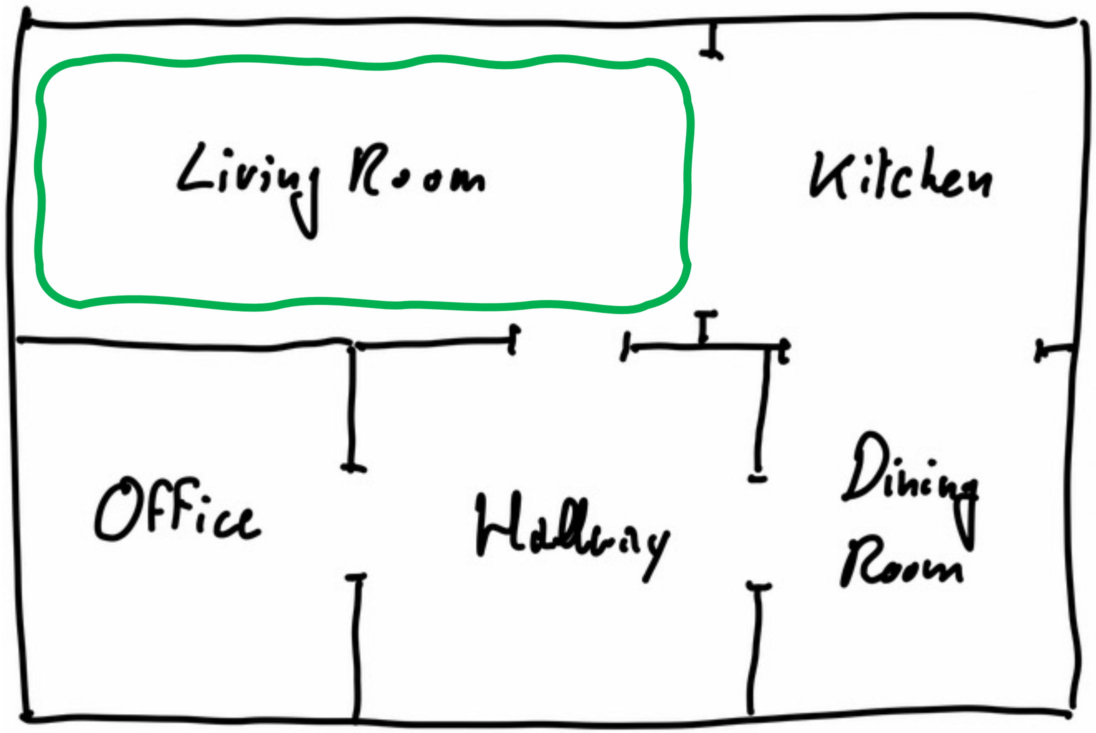
$$\mathsf{E}\left[U\left(a_{1,} \ldots, a_n, x_1, X_2, \ldots X_{n+1}\right)\right] = \mathsf{E}[R(x_1, a_1, X_2) + \gamma R(X_2, a_2, X_3) + \cdots \gamma^{n-1} R(X_n, a_n, X_{n+1})\ ]$$

- Again, we can compute the expectation to choose between finite horizon plans
- For n=3, we have $4^3 = 64$ possible plans, and for each plan we must evaluate $5^4 = 625$ possible state sequences
- An approximate algorithm to evaluate a given plan:
  - Simulate multiple rollouts
  - Average the result
- Still expensive, only practical for short horizon plans…

Policies $\pi: \mathcal{X} \rightarrow \mathcal{A}$

- Because actions are non-deterministic, fixed plans are brittle and prone to failure.
- Better to have a *state-dependent* plan
- A policy $\pi(X)$ is a function that specifies which action to take in each state.
- Let us come up with a policy together:

  - $\pi(L) =$
  - $\pi(K) =$
  - $\pi(O) =$
  - $\pi(H) =$
  - $\pi(D) =$



The Value Function for a Policy

- Recall the Expected Utility

$$\bar{U}(a_1 \ldots a_n, x_1) = E\left[\sum_{i=1}^{n} \gamma^{i-1} R(X_i, a_i, X_{i+1})\right]$$

- For a policy, we can define this similarly:

$$\bar{U}(\pi, n, x_1) \doteq E\left[R(x_1, \pi(x_1), X_2) + \gamma\, R(X_2, \pi(X_2), X_3) + \cdots + \gamma^2 R(X_n, \pi(X_n), X_n)\right]$$

- Can be extended to infinite horizon policy, defining the value function:

$$V^{\pi}(x_1) \doteq E\left[R(x_1, \pi(x_1), X_2) + \gamma\, R(X_2, \pi(X_2), X_3) + \gamma^2 R(X_3, \pi(X_3), X_4) + \cdots\right]$$

- Of course, above holds for arbitrary x_t, not just x_1.

Recursive Definition of V^{π}

$$V^{\pi}(x_1) = E[R(x_1, \pi(x_1), X_2) + \gamma R(X_2, \pi(X_2), X_3) + \gamma^2 R(X_3, \pi(X_3), X_4) + \ \ldots]$$

$$V^{\pi}(x_1) = \sum_{x_2} P(x_2 | x_1, \pi(x_1)) \{R(x_1, \pi(x_1), x_2) + \gamma E[R(x_2, \pi(x_2), X_3) + \gamma R(X_3, \pi(X_3), X_4) + \ \ldots]\}$$

$$V^{\pi}(x_1) = \sum_{x_2} P(x_2 | x_1, \pi(x_1)) \{R(x_1, \pi(x_1), x_2) + \gamma V^{\pi}(x_2)\}$$

$$V^{\pi}(x_1) = \sum_{x_2} P(x_2 | x_1, \pi(x_1)) R(x_1, \pi(x_1), x_2) \ + \gamma \sum_{x_2} P(x_2 | x_1, \pi(x_1)) V^{\pi}(x_2)$$

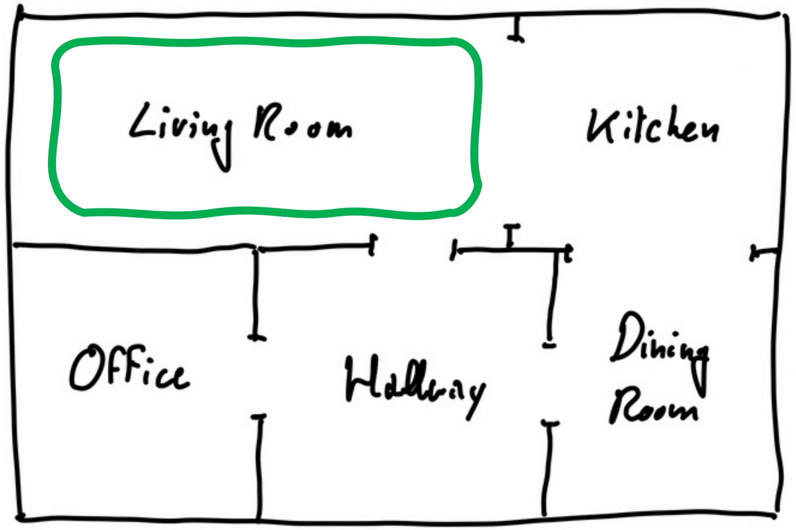
$$V^{\pi}(x) = \bar{R}(x, \pi(x)) \ + \gamma \sum_{x'} P(x' | x, \pi(x)) V^{\pi}(x')$$

Exact Computation for V^π

- Because we have a finite set of states, we get 5 linear equations in 5 unknowns $V^\pi(x)$:

$$V^\pi(x) = \bar{R}(x, \pi(x)) + \gamma \sum_{x'} P(x'|x, \pi(x)) V^\pi(x')$$

- Can be solved efficiently with *np.linalg.solve*
- Example in book:




```
reasonable_policy = [UP, LEFT, RIGHT, UP, LEFT]
```

```
[[ 0.1   -0.    -0.    -0.    -0.  ]        [[10.]
 [-0.72   0.82 -0.    -0.    -0.  ]         [ 8.]
 [-0.    -0.     0.82 -0.72 -0.  ]         [ 0.]
 [-0.72 -0.    -0.     0.82 -0.  ]         [ 8.]
 [-0.    -0.    -0.    -0.72  0.82]]        [ 0.]]
```

```
V(reasonable_policy):
  Living Room : 100.00
  Kitchen     : 97.56
  Office      : 85.66
  Hallway     : 97.56
  Dining Room : 85.66
```

Policy Iteration

Start with a random policy π^0, and repeat until convergence:

1. Compute the value function V^{π^k}
2. Improve the policy for each state x using the update rule:

$$\boldsymbol{\pi^{k+1}}(\boldsymbol{x}) \leftarrow \boldsymbol{\arg\max_{a}} \left\{ \overline{\boldsymbol{R}}(\boldsymbol{x}, \boldsymbol{a}) + \boldsymbol{\gamma} \sum_{\boldsymbol{x'}} \boldsymbol{P}(\boldsymbol{x'} | \boldsymbol{x}, \boldsymbol{a}) \boldsymbol{V}^{\boldsymbol{\pi^k}}(\boldsymbol{x'}) \right\}$$


```
always_right = [RIGHT, RIGHT, RIGHT, RIGHT, RIGHT]
```
✓ 0.7s

```
optimal_policy, optimal_value_function = policy_iteration(always_right)
print([vacuum.action_space[a] for a in optimal_policy])
```
✓ 0.7s

```
['L', 'L', 'R', 'U', 'U']
```

Optimal Value Function

The optimal value function is the one corresponding to the optimal policy:

$$
\begin{aligned}
V^*(x) &= \max_{\pi} V^{\pi}(x) \\
&= \max_{\pi}\left\{\bar{R}(x,\pi(x)) \ +\gamma\sum_{x'} P(x'|x,\pi(x))V^{\pi}(x')\right\} \\
&= \max_{a}\left\{\bar{R}(x,a) \ +\gamma\sum_{x'} P(x'|x,a))V^{*}(x')\right\}
\end{aligned}
$$

The Bellman equation:

$$
\boldsymbol{V^*(x) = \max_{a}\left\{\bar{R}(x,a) \ +\gamma\sum_{x'} P(x'|x,a))V^{*}(x')\right\}}
$$

Value Iteration

Start with a random value function V^0, and repeat until convergence:

- Improve the value function V^k using the update rule:

$$\boldsymbol{V^{k+1}(x) \leftarrow \max_{a}\left\{\bar{R}(x,a) + \gamma \sum_{x'} P(x'|x,a)) V^k(x')\right\}}$$

```
V_k = np.full((5,), 100)
for k in range(10):
    Q_k = np.sum(T * (R + 0.9 * V_k), axis=2) # 5 x 4
    V_k = np.max(Q_k, axis=1) # max over actions
    print(np.round(V_k,2))
```

✓ 0.4s

```
[100.  98.  90.  98.  90.]
[100.    97.64  86.76  97.64  86.76]
[100.    97.58  85.92  97.58  85.92]
[100.    97.56  85.72  97.56  85.72]
```

Optimal Policy

Given the $\boldsymbol{V^*(x)}$, computing the optimal policy is a straightforward optimization:

$$\boldsymbol{\pi^*(x) = \arg\max_a \left\{ \bar{R}(x,a) + \gamma \sum_{x'} P(x'|x,a)) V^*(x') \right\}}$$

For convenience, we define the $\boldsymbol{Q^*}$ function as

$$\boldsymbol{Q^*(x,a) = \bar{R}(x,a) + \gamma \sum_{x'} P(x'|x,a)) V^{\pi}(x')}$$

and we can write the optimal policy as:

$$\boldsymbol{\pi^*(x) = \arg\max_a Q^*(x,a)}$$

The Q function plays a role in reinforcement learning, to be continued…