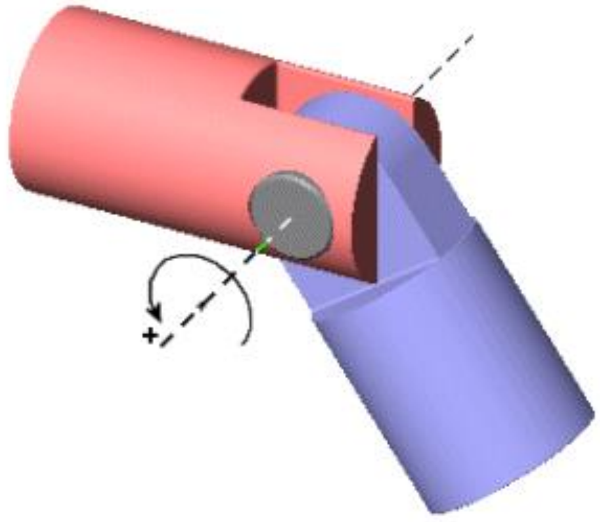
CS 3630

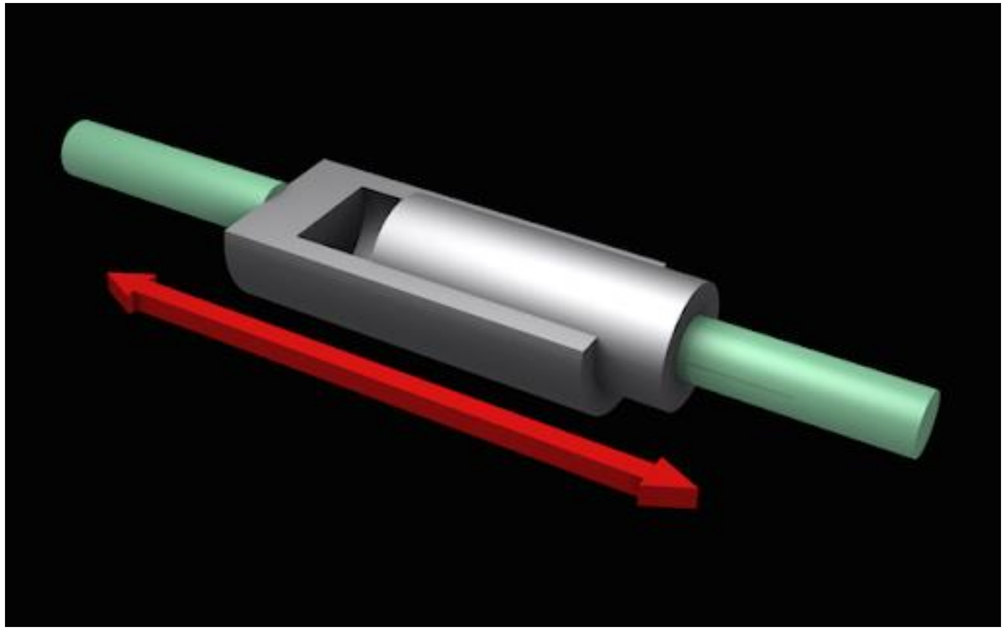
Robot Kinematics:
Planar Arms

Robot Arms

- A robot arm (aka serial link manipulator) consists of a series of rigid links, connected by joints (motors), each of which has a single degree of freedom.
  - Revolute Joint: Single degree of freedom is rotation about an axis.
  - Prismatic joint: Single degree of freedom is translation along an axis.



Revolute Joint

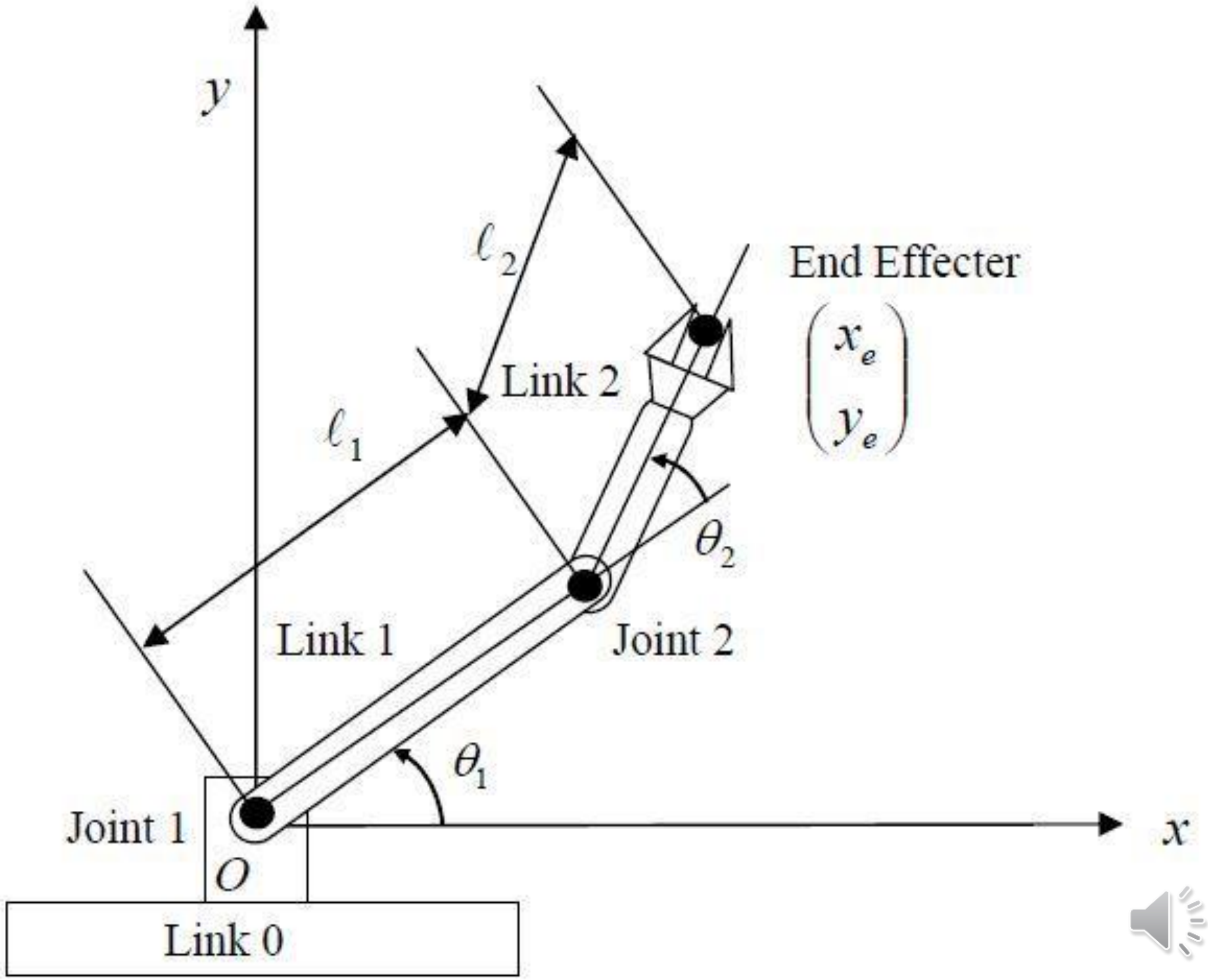


Prismatic Joint

Describing Serial Link Arms

- Number the links in sequence.
- For a robot with n joints:
  - Base (which does not move) is Link 0.
  - End-effector (tool) is attached to Link n.
  - Joint i connects Link $i-1$ to Link i
  - We define the joint variable q_i for joint i as:

$$q_i = \begin{cases} \theta_i \text{ if joint } i \text{ is revolute} \\ d_i \text{ if joint } i \text{ is prismatic} \end{cases}$$



Two-link Planar Arm:

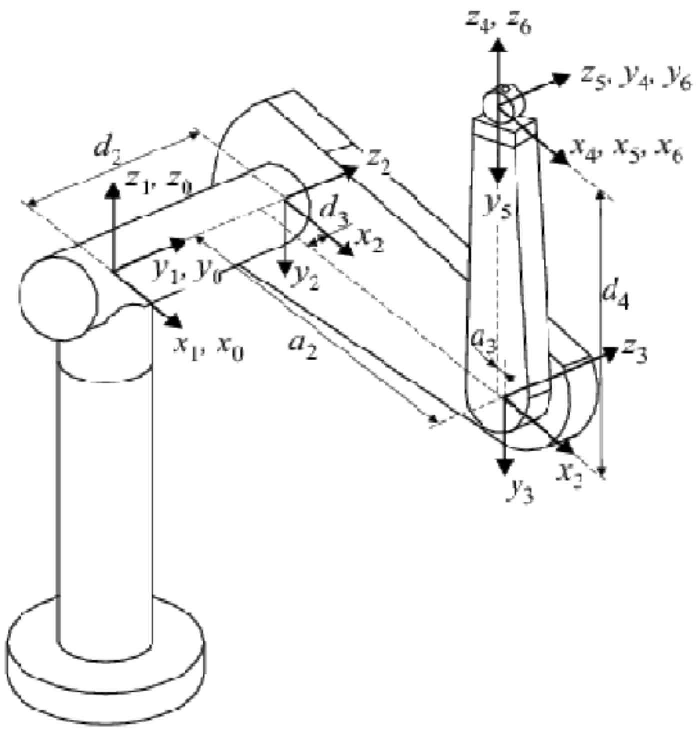
- $n = 2$,
- both links are always coplanar (no rotation out of the plane).
- $q_1 = \theta_1, \; q_2 = \theta_2$

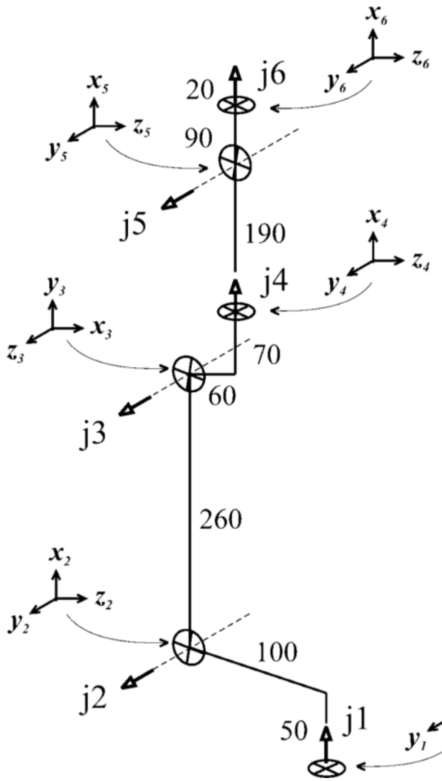
Manipulator Kinematics

- Kinematics describes the position and motion of a robot, without considering the forces required to cause the motion.
- Forward Kinematics: *Given the value for each joint variable, q_i, determine the position and orientation of the end-effector (gripper, tool) frame.*

<u>The basic idea:</u>

➢ Assign lots of coordinate frames, and express these frames in terms of the joint variables, q_i.





General Approach

- Each link is a rigid body.
- We know how to describe the position and orientation of a rigid body:
  - Attach a coordinate frame to the body.
  - Specify the position and orientation of the coordinate frame relative to some reference frame.
- If two links, say link $i-1$ and link i are connected by a single joint, then the relationship between the two frames can be described by a homogeneous transformation matrix T_i^{i-1} which *will depend only on the value of the joint variable*!

Homogeneous Transformations

We can simplify the equation for coordinate transformations by augmenting the vectors and matrices with an extra row:

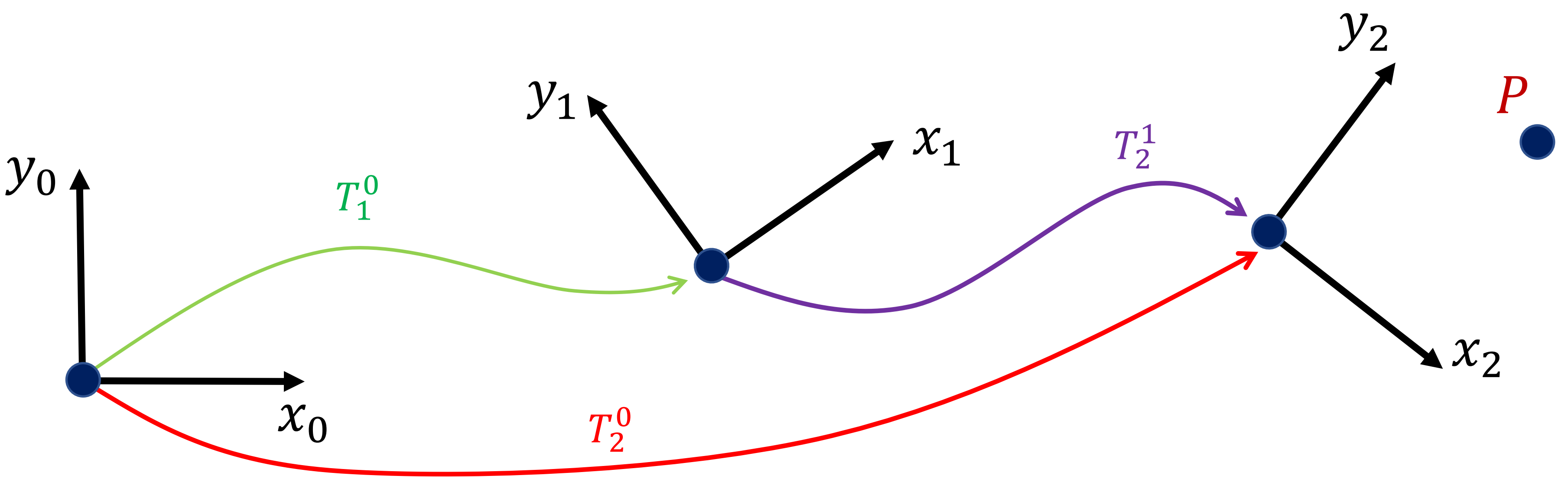
$$\begin{bmatrix} \boldsymbol{P}^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^0 \boldsymbol{P}^1 + \boldsymbol{d}^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{d}^0 \\ 0_2 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^1 \\ 1 \end{bmatrix}$$

$$\tilde{P}^0 = \begin{bmatrix} \boldsymbol{P}^0 \\ 1 \end{bmatrix}, \tilde{P}^1 = \begin{bmatrix} \boldsymbol{P}^1 \\ 1 \end{bmatrix}$$

$$\tilde{P}^0 = T_1^0 \tilde{P}^1$$

- $\mathbf{T}_1^0$ **is called a homogeneous transformation matrix**
- $\widetilde{\mathbf{P}}^0$ **are the homogeneous coordinates for** $\mathbf{P}^0$

Composition of Transformations



From our previous results, we know:

$$\tilde{P}^0 = T_1^0 \tilde{P}^1$$
$$\tilde{P}^1 = T_2^1 \tilde{P}^2$$

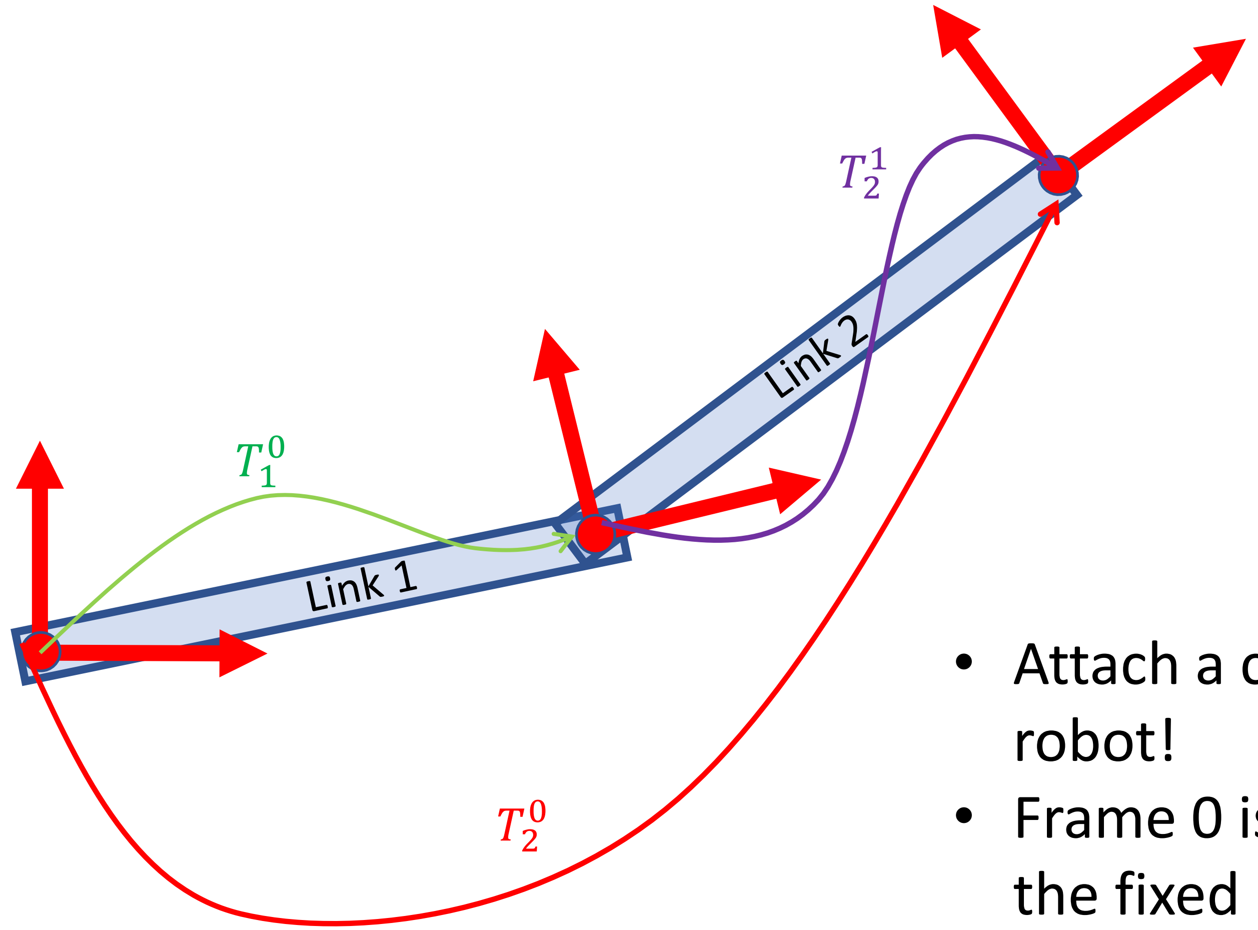
$$\Rightarrow \quad \tilde{P}^0 = T_1^0 T_2^1 \tilde{P}^2$$

But we also know: $\tilde{P}^0 = T_2^0 \tilde{P}^2$

$$\Rightarrow \quad T_2^0 = T_1^0 T_2^1$$

This is the composition law for homogeneous transformations.

What about robot arms??

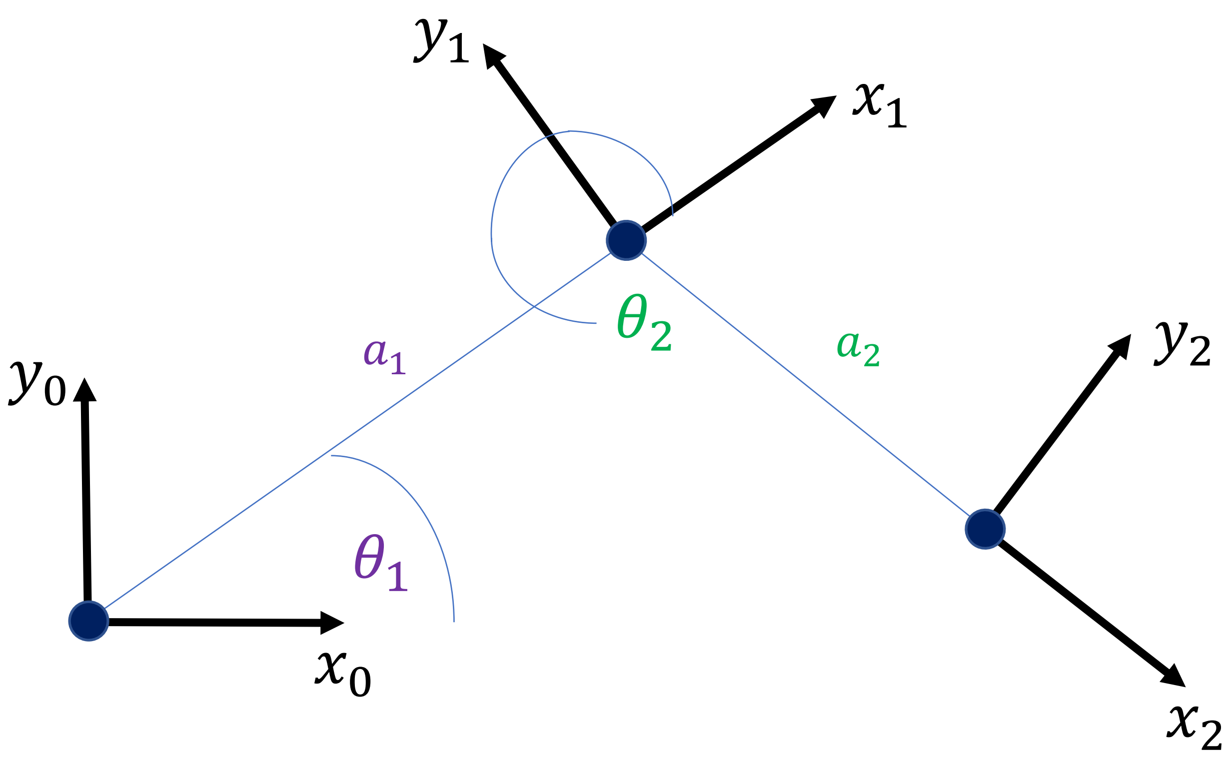


- Attach a coordinate frame to each link of the robot!
- Frame 0 is attached to Link 0, which is merely the fixed mounting point to the environment.
- Now, the trick is to express T_i^{i-1} as a function of θ_i

A special case

Suppose the axis x_i is collinear with the origin of Frame $i-1$:

- x_1 is collinear with the origin of Frame 0
- x_2 is collinear with the origin of Frame 1



$$T_1^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & a_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & a_1\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & a_2\sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Use this to simplify link coordinate frames!

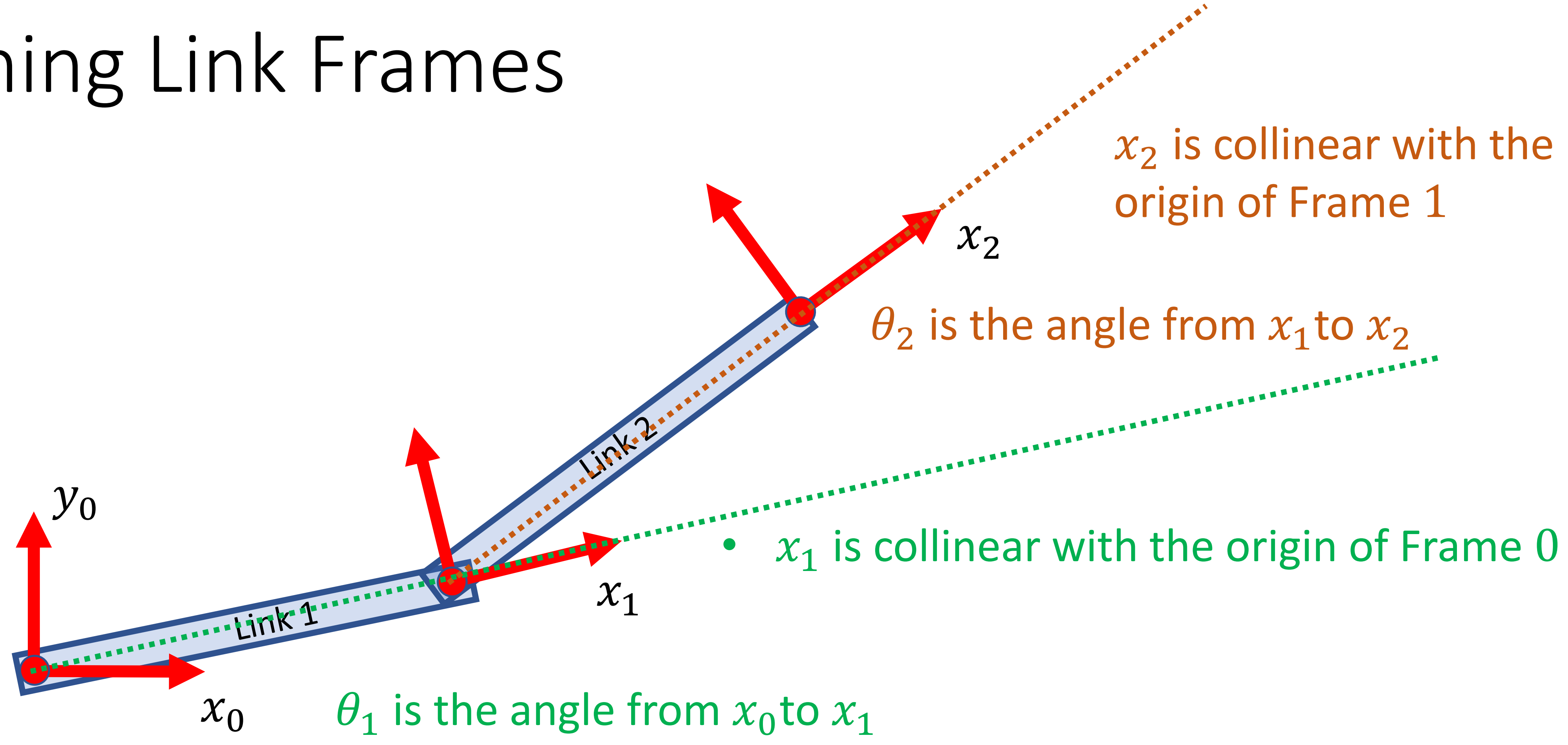
$$T_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a_i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i & a_i\sin\theta_i \\ 0 & 0 & 1 \end{bmatrix}$$

Assigning Coordinate Frames to Links

- Frame 0 (the base frame) has its origin at the center of Joint 1 (on the axis of rotation).
- Frame i is ***rigidly attached*** to Link i, and has it's origin at the center of Joint $i+1$.
- The x_i-axis is collinear with the origin of Frame $i-1$.
- The link length, a_i is the distance between the origins of Frames i and $i-1$.
- The homogeneous transformation that relates adjacent frames is given by:

$$T_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i & a_i\sin\theta_i \\ 0 & 0 & 1 \end{bmatrix}$$

Assigning Link Frames



- Frame n is the end-effector frame. It can be attached to link n in any manner that is convenient.
- In this case, $n = 2$, so Frame 2 is the end-effector frame.

The Forward Kinematic Map

- The forward kinematic map gives the position and orientation of the end-effector frame as a function of the joint variables:

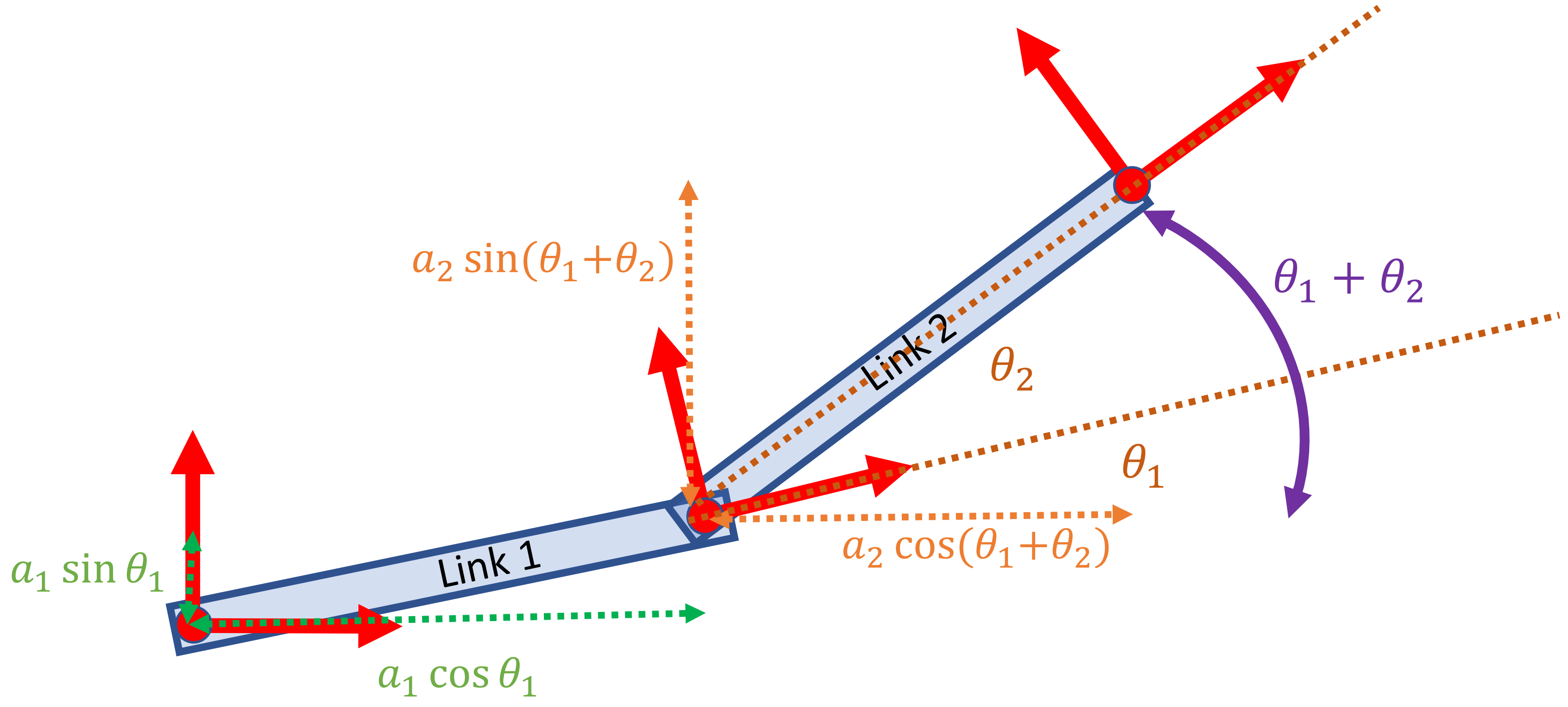
$$T_n^0 = F(q_1, \dots, q_n)$$

- For the two-link planar arm, we have

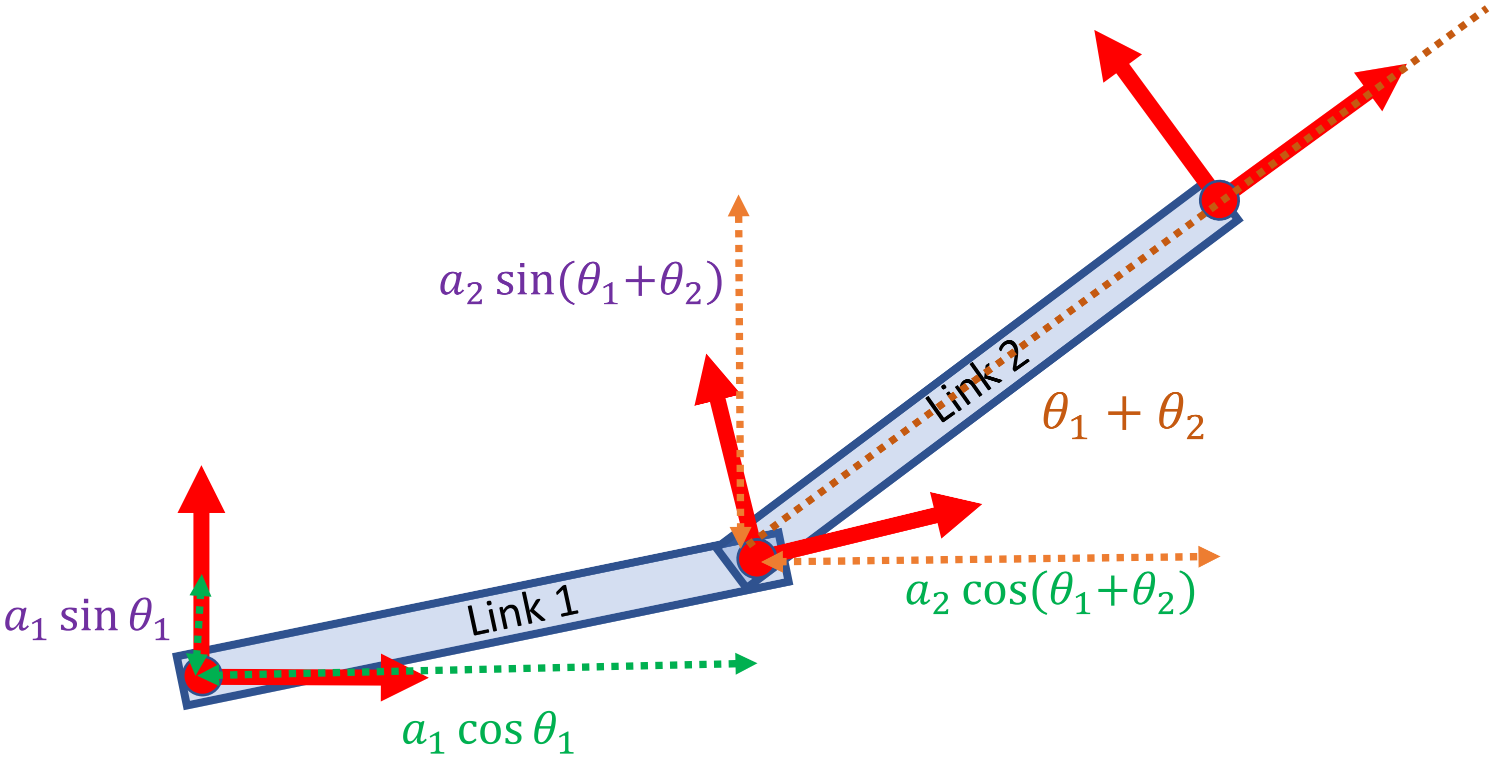
$$T_2^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & a_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & a_1\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & a_2\sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

Simple Geometry…



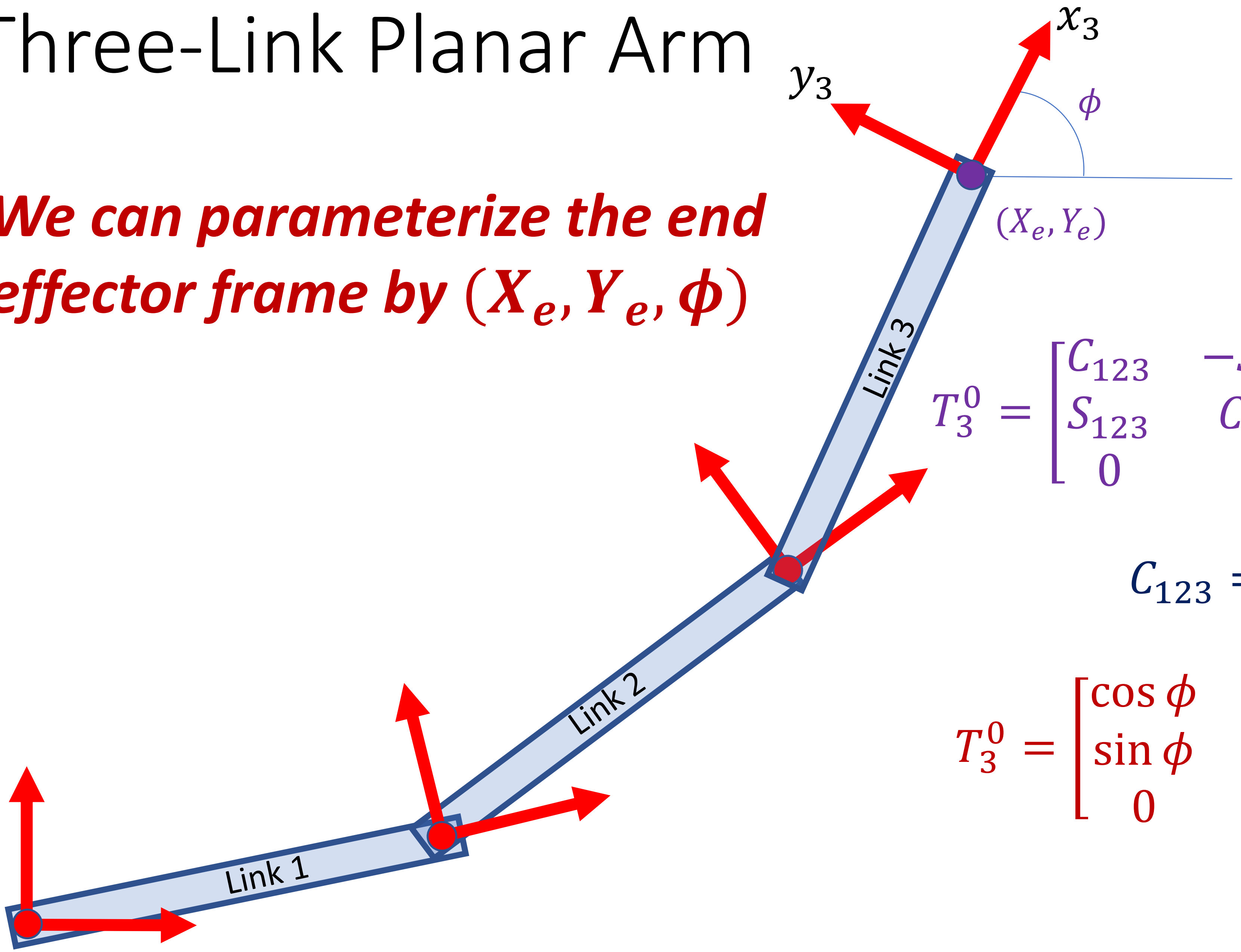
Simple Geometry…



$$T_2^0 = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

Three-Link Planar Arm

We can parameterize the end effector frame by (X_e, Y_e, ϕ)



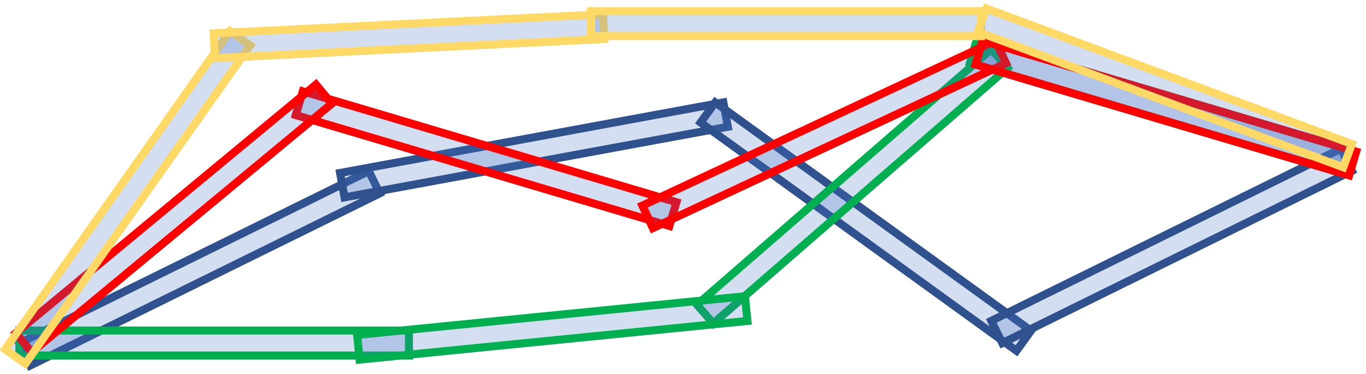
$$T_3^0 = \begin{bmatrix} C_{123} & -S_{123} & a_1C_1 + a_2C_{12} + a_3C_{123} \\ S_{123} & C_{123} & a_1S_1 + a_2S_{12} + a_3S_{123} \\ 0 & 0 & 1 \end{bmatrix}$$

$C_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$, etc.

$$T_3^0 = \begin{bmatrix} \cos\phi & -\sin\phi & X_e \\ \sin\phi & \cos\phi & Y_e \\ 0 & 0 & 1 \end{bmatrix}$$

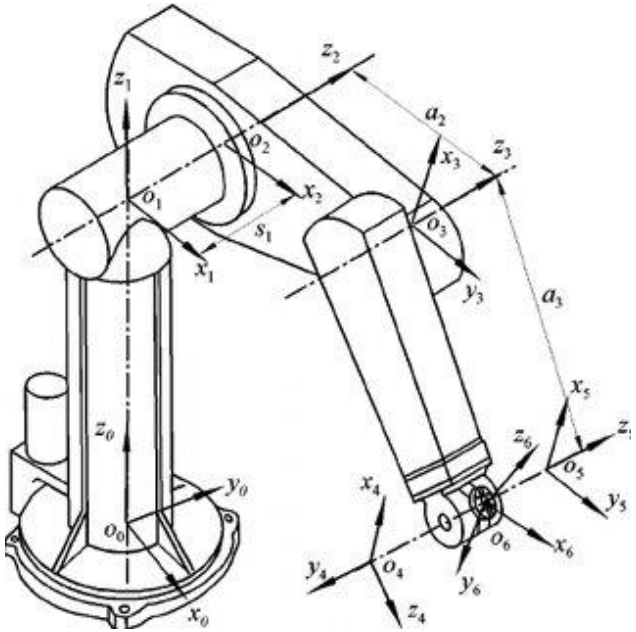
About the Forward Kinematic Map

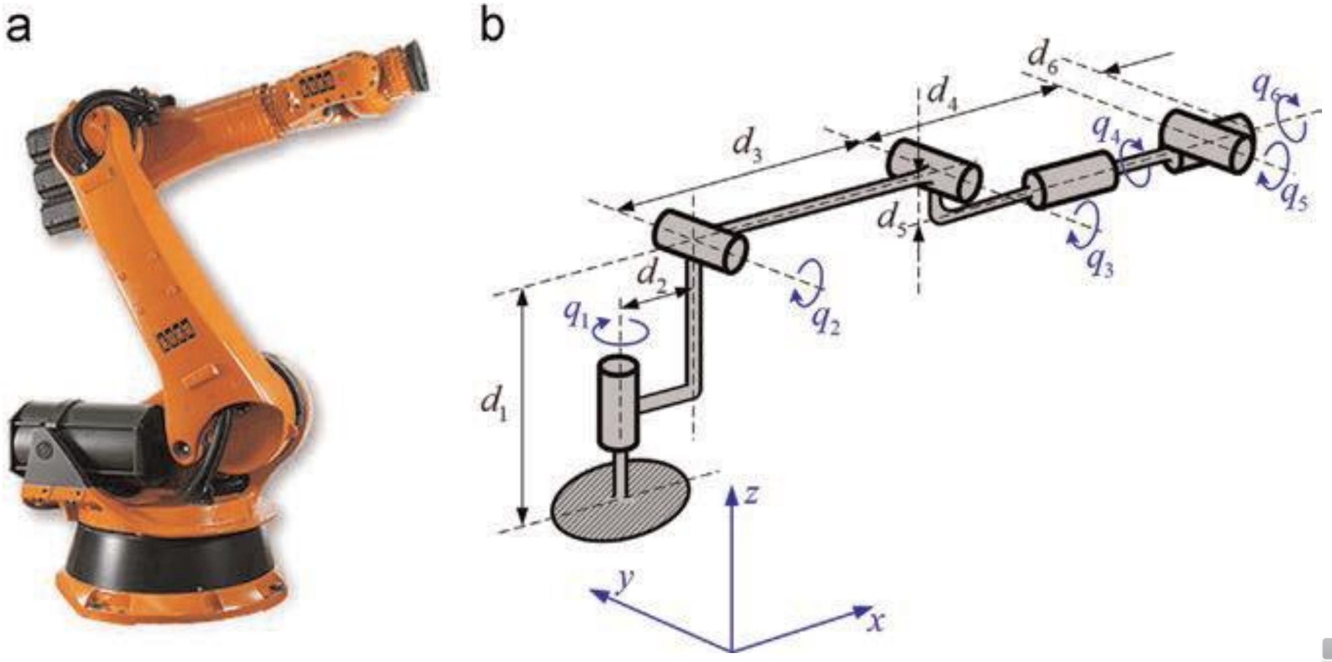
- For the two-link arm, we can **position** the end-effector origin anywhere in the arm's workspace: two inputs (θ_1, θ_2) and two "outputs" (X_e, Y_e).
- For the three-link arm, we can position the end-effector origin anywhere in the arm's workspace, ***and*** we can choose the orientation of the frame: three inputs $(\theta_1, \theta_2, \theta_3)$ and three "outputs" (X_e, Y_e, ϕ).
- Suppose we had a four-link arm?
  - Infinitely may ways to achieve a desired end-effector configuration (X_e, Y_e, ϕ).



More General Robot Arms

- With a bit of work, this can be generalized to arbitrary robot arms.
- We shall not do this bit of work in CS3630.





Motion Control

- Trajectory following is important
  - Spray-painting
  - Sealing
  - Welding
- Three main approaches:
  - Trajectory replay
  - Joint-space Motion Control
  - Cartesian Motion Control

Image by Roboguru

Trajectory Replay

- Teaching by demonstration
- Define a set of **waypoints** by “showing” the robot
- Similar to keyframe animation in graphics
- Still need to interpolate between waypoints

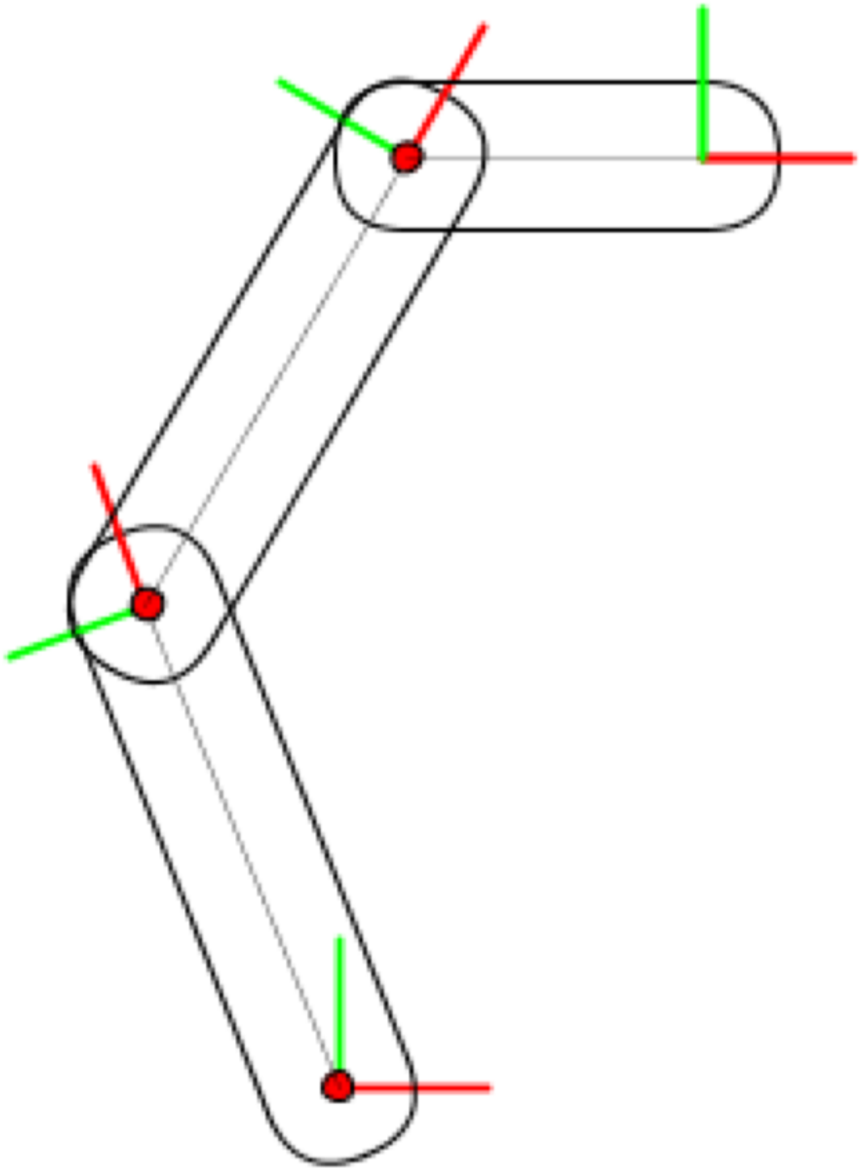
RRR example

$$T_1^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 3.5\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 3.5\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 3.5\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 3.5\sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_3^2 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 2\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 2\sin\theta_3 \\ 0 & 0 & 1 \end{bmatrix}$$

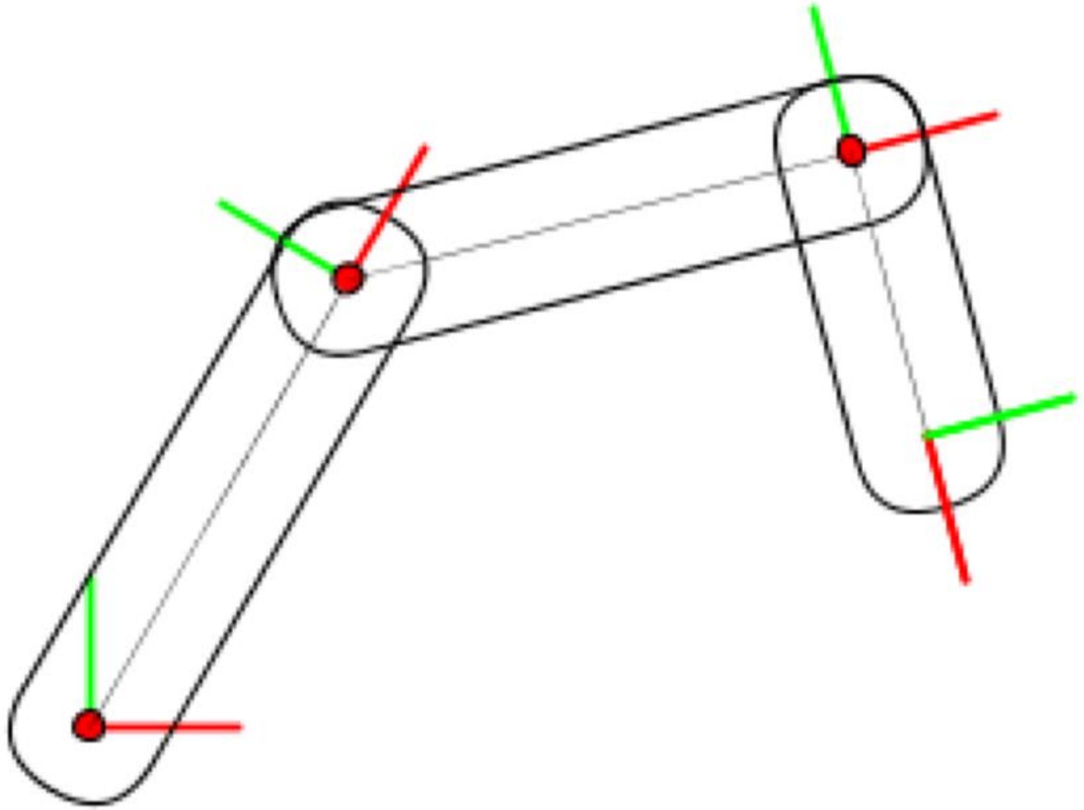
- End-effector == frame 3



(a) $\theta_1 = 112^{\circ}$, $\theta_2 = -52^{\circ}$, and $\theta_3 = -60^{\circ}$

RRR example, cont'd

- Multiply 3 matrices
- Note R in upper left
- Check orientation!

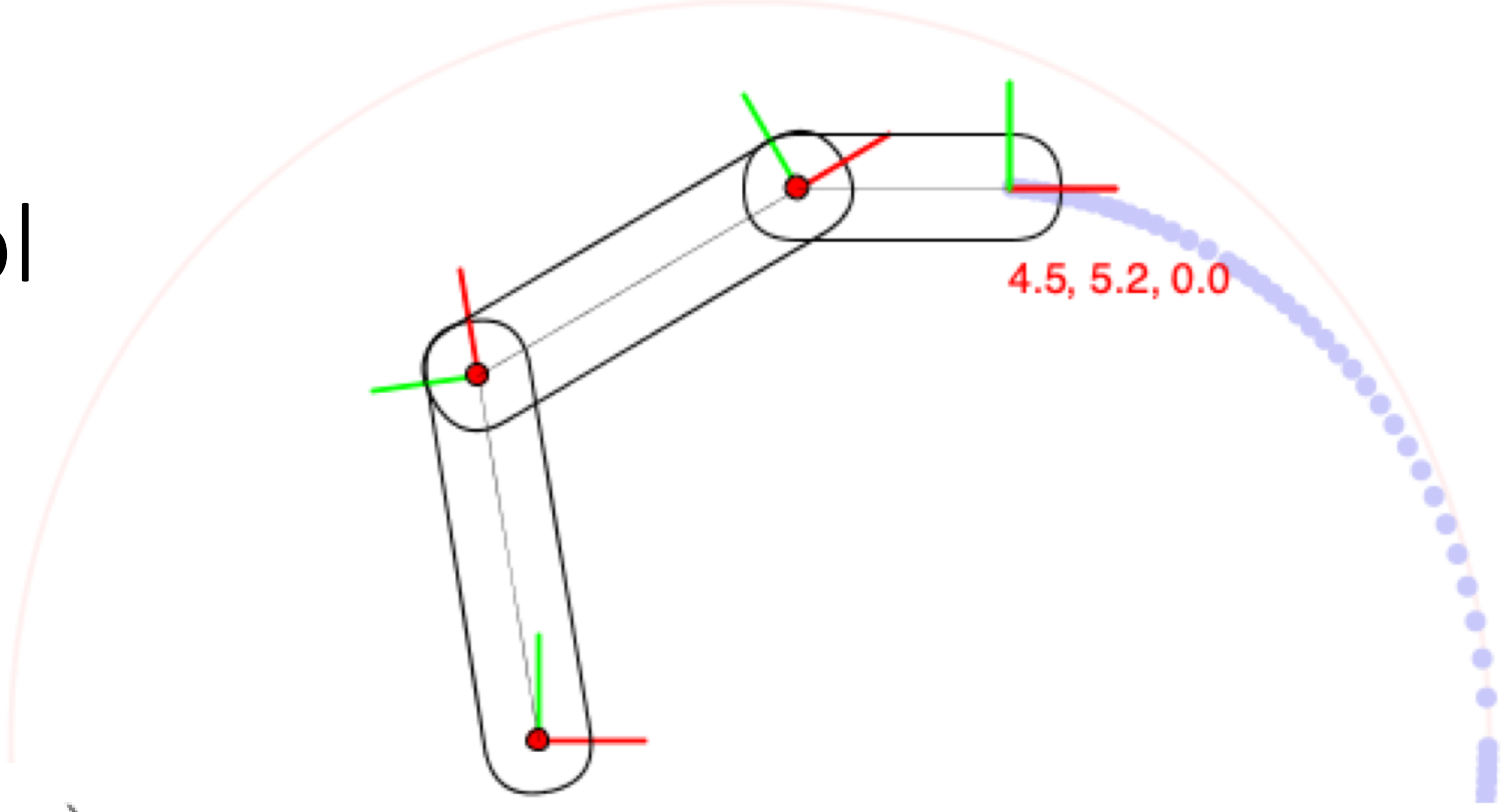


(b) $\theta_1 = 60^o$, $\theta_2 = -45^o$, and $\theta_3 = -90^o$

$$T_t^s(q) = \begin{pmatrix} \cos\beta & -\sin\beta & 3.5\cos\theta_1 + 3.5\cos\alpha + 2\cos\beta \\ \sin\beta & \cos\beta & 3.5\sin\theta_1 + 3.5\sin\alpha + 2\sin\beta \\ 0 & 0 & 1 \end{pmatrix}$$

with $\alpha = \theta_1 + \theta_2$ and $\beta = \theta_1 + \theta_2 + \theta_3$, the latter being the tool orientation.

Proportional Feedback Control

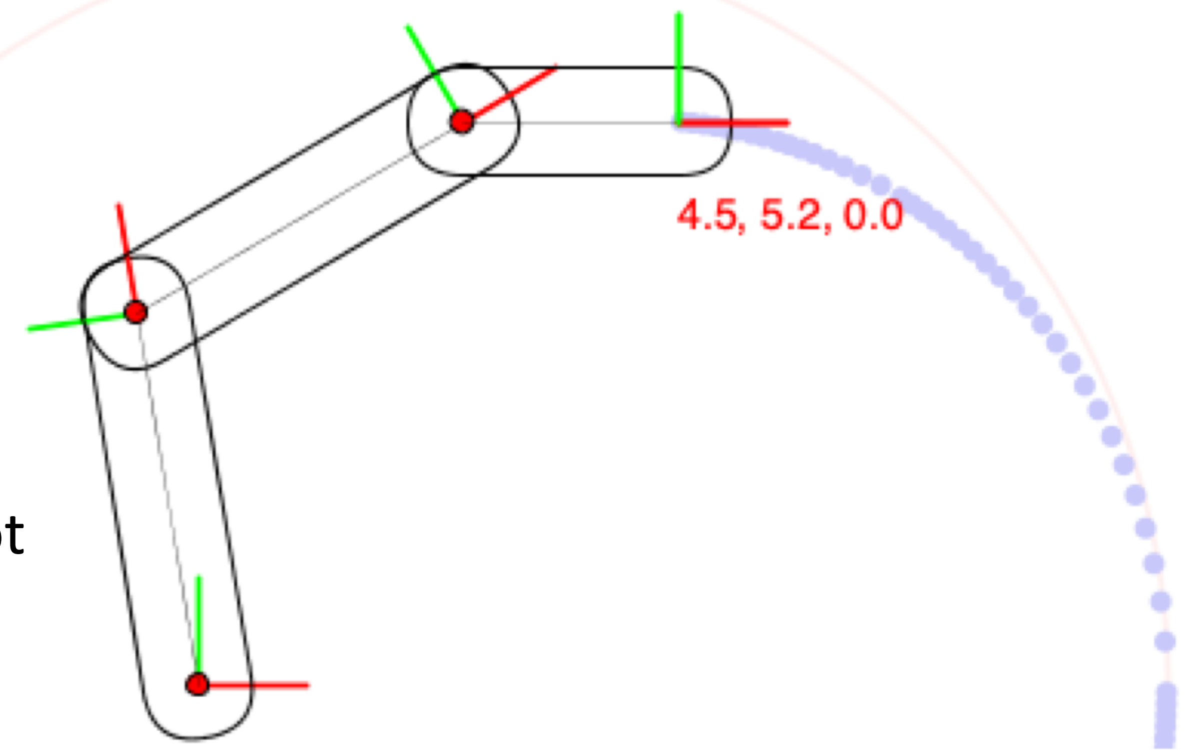


- Feedback law:

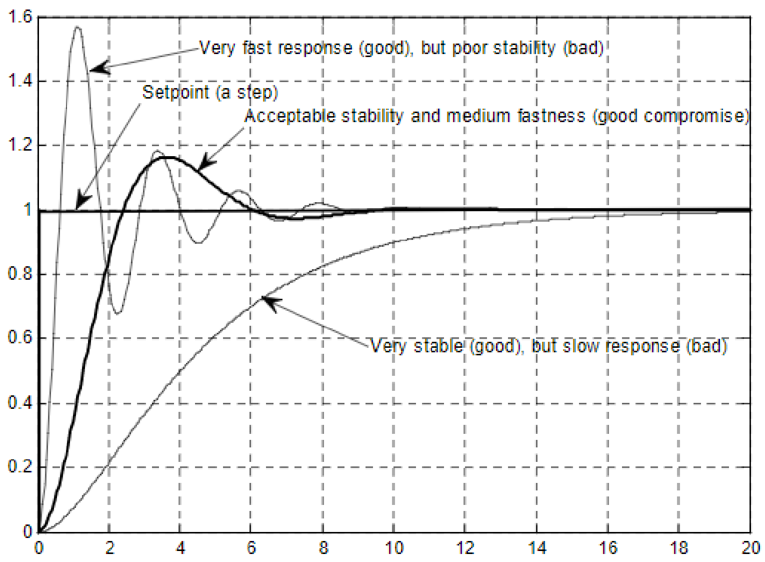
$$q_{t+1} = q_t + K_p(q_d - q_t)$$

- At every time step:
  - Calculate **joint space error** $e_t = q_d - q_t$
  - Increase of decrease proportional to e_t
  - K_p is proportional gain parameter

Proportional Feedback Control



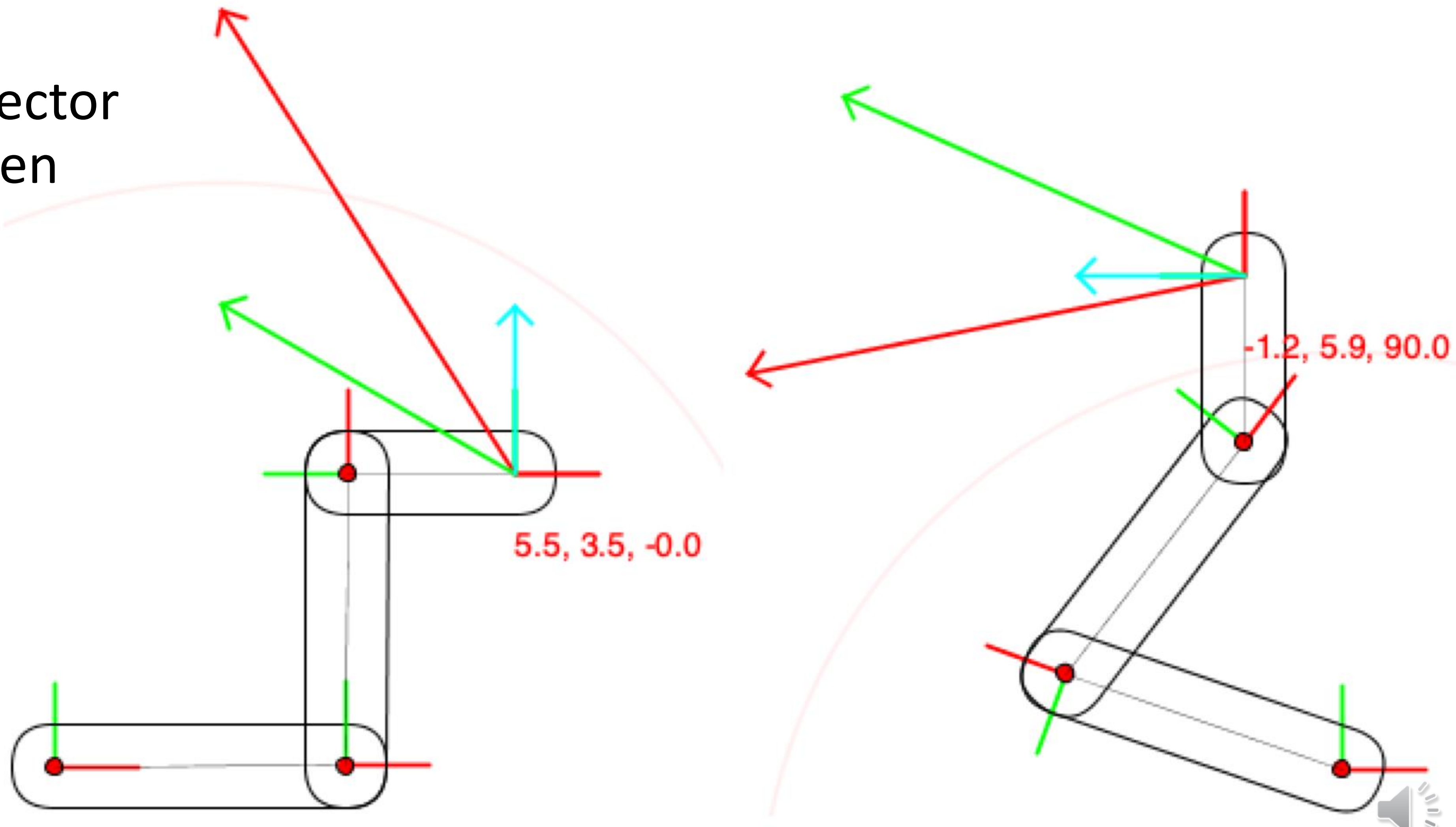
- Properties:
  - Closer to goal -> smaller steps
  - Automatically reverses sign if we overshoot
  - Generalizes to vector-valued control
- Value of Kp really matters:
  - too high: overshoot
  - too low: slow convergence
- Special case of PID control



https://arduinoplusplus.wordpress.com/2017/06/21/pid-control-experiment-tuning-the-controller/

The Manipulator Jacobian

- Velocity of end-effector if we move any given joint?
- Given by arrows:
- R=joint 1
- G=joint 2
- B=joint 3



Jacobian = linear map

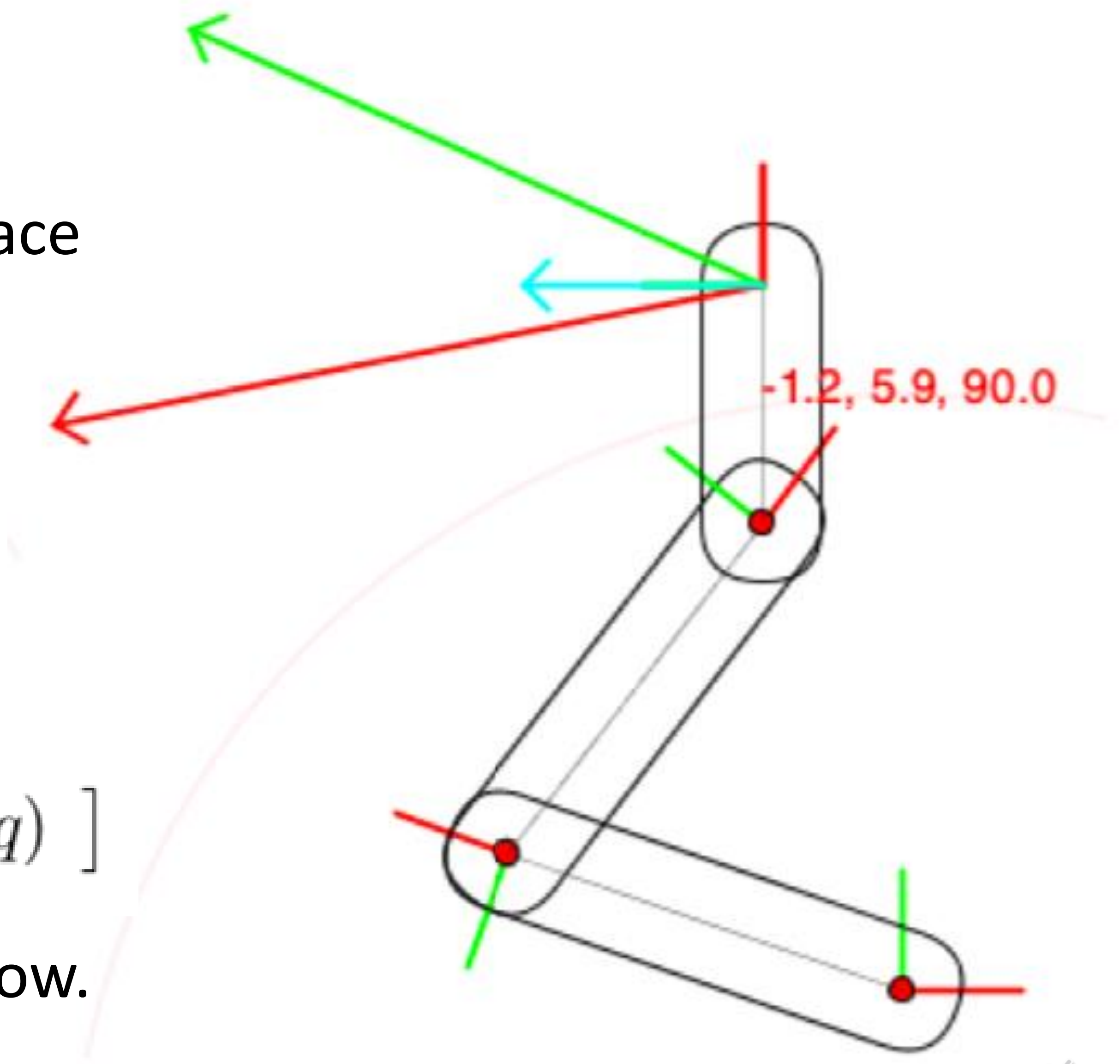
- Linear relationship between joint space velocity and cartesian velocity (pose space!)

$$[\dot{x}, \dot{y}, \dot{\theta}]^T = J(q)\dot{q}$$

- J is *3xn* matrix:

$$J(q) \triangleq \begin{bmatrix} J_1(q) & J_2(q) & \dots & J_n(q) \end{bmatrix}$$

- Each $J_i(q)$ column corresponds to arrow.
- Partial derivative of pose wrt q_i



Worked Example: RRR manipulator

- Remember:

$$T_t^s(q) = \begin{pmatrix} \cos\beta & -\sin\beta & 3.5\cos\theta_1 + 3.5\cos\alpha + 2\cos\beta \\ \sin\beta & \cos\beta & 3.5\sin\theta_1 + 3.5\sin\alpha + 2\sin\beta \\ 0 & 0 & 1 \end{pmatrix}$$

- Extracting x, y, theta:

$$\begin{bmatrix} x(q) \\ y(q) \\ \theta(q) \end{bmatrix} = \begin{bmatrix} 3.5\cos\theta_1 + 3.5\cos\alpha + 2\cos\beta \\ 3.5\sin\theta_1 + 3.5\sin\alpha + 2\sin\beta \\ \beta \end{bmatrix}$$

- So what is Jacobian???

Worked Example: RRR manipulator

- x, y, theta:

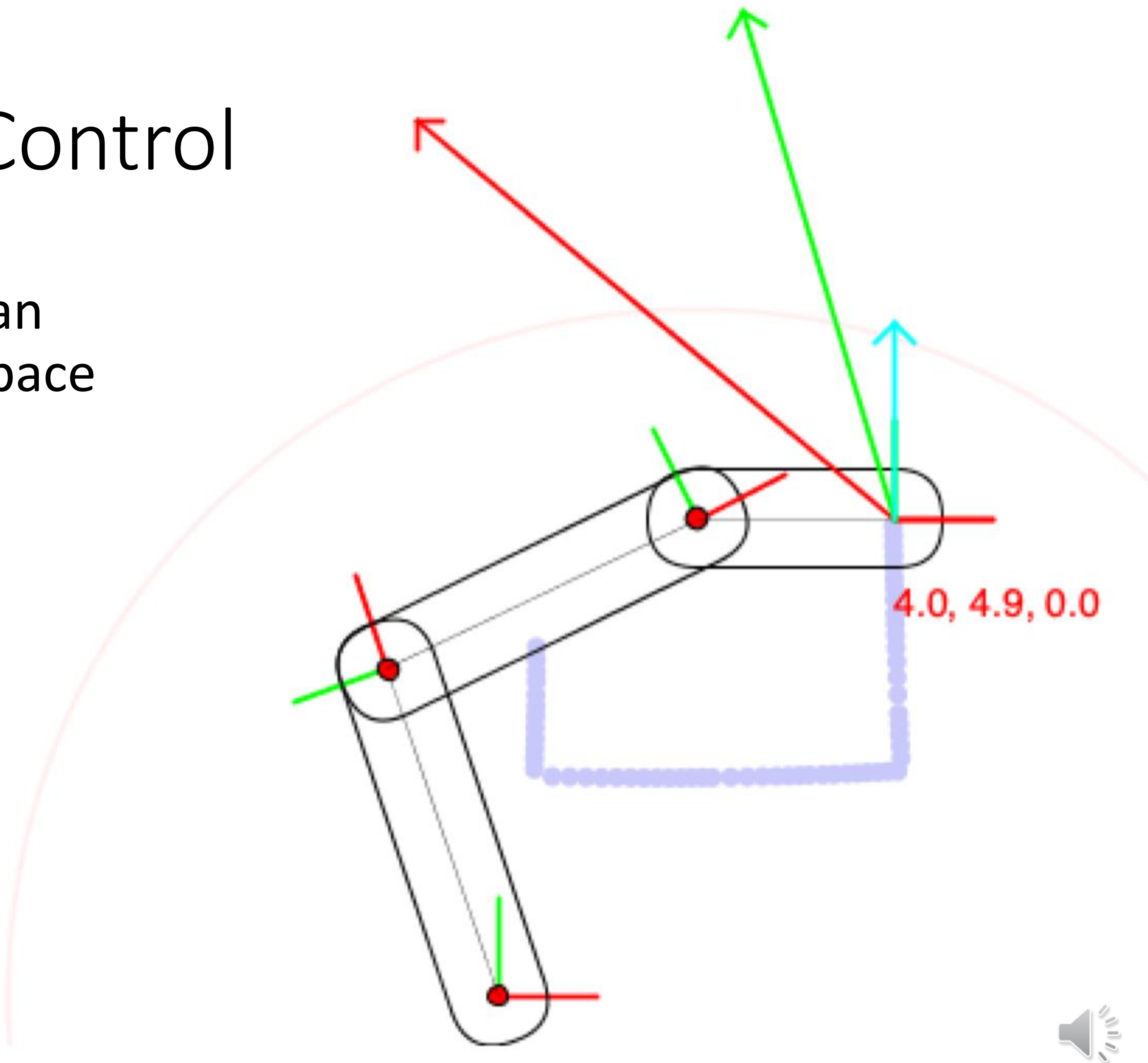
$$\begin{bmatrix} x(q) \\ y(q) \\ \theta(q) \end{bmatrix} = \begin{bmatrix} 3.5\cos\theta_1 + 3.5\cos\alpha + 2\cos\beta \\ 3.5\sin\theta_1 + 3.5\sin\alpha + 2\sin\beta \\ \beta \end{bmatrix}$$

- Jacobian:

$$\begin{pmatrix} -3.5\sin\theta_1 - 3.5\sin\alpha - 2.5\sin\beta & -3.5\sin\alpha - 2.5\sin\beta & -2\sin\beta \\ 3.5\cos\theta_1 + 3.5\cos\alpha + 2.5\cos\beta & 3.5\cos\alpha + 2.5\cos\beta & 2\cos\beta \\ 1 & 1 & 1 \end{pmatrix}$$

Cartesian Motion Control

- Convert direction in cartesian space to direction in joint space
- Yields straight-line paths



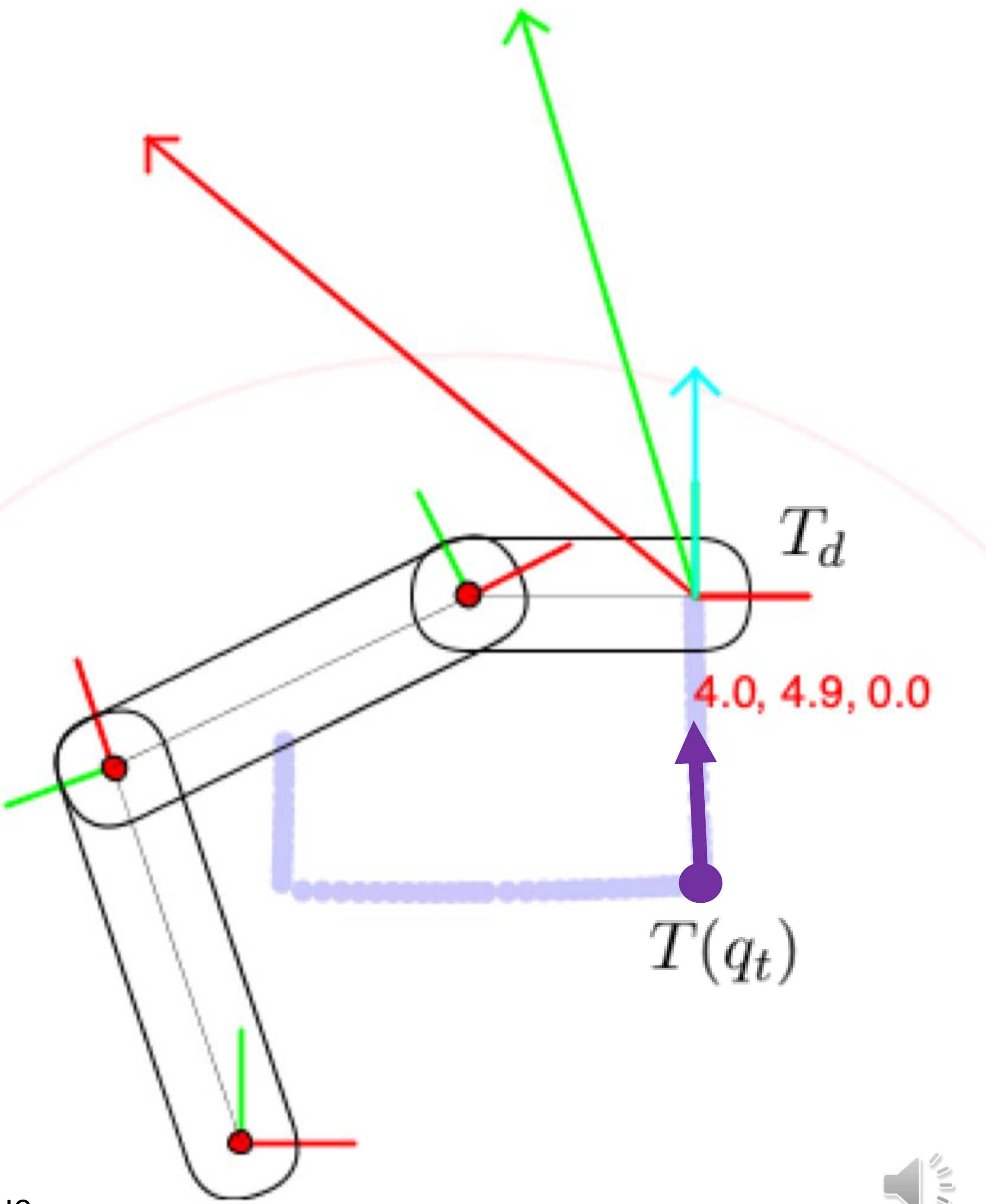
How do we convert?

- We want a straight line!
- Calculate (scaled) direction of the line
- Error in cartesian space:

$$E_t(q) = \begin{bmatrix} e_x \\ e_y \\ e_\theta \end{bmatrix} = \begin{bmatrix} x_d - x(q_t) \\ y_d - y(q_t) \\ \theta_d \ominus \theta(q_t) \end{bmatrix}$$

- Then, simple proportional control:

$$q_{t+1} = q_t + K_p J(q_t)^{-1} E_t(q)$$



Small print: we have to take care when subtracting angles, as they are not unique

Summary

1. **Forward Kinematics** is just multiplying transforms
2. We went through an **RRR Worked Example**
3. **Joint-Space Motion Control** creates paths that minimize distance in joint space
4. The **Manipulator Jacobian** provides a relationship between cartesian and joint-space velocities/displacements
5. **Cartesian Motion Control** exploits this relationship to provide predictable paths in cartesian space