Outline

• Intro
• Camera Review
• Stereo triangulation
• Geometry of 2 views
  – Essential Matrix
  – Fundamental Matrix
• Estimating E/F from point-matches
Why Consider Multiple Views?

Answer: To extract 3D structure via triangulation.
Stereo Rig

Top View

Matches on Scanlines

Convenient when searching for correspondences.
Feature Matching!
Real World Challenges

Bad News: Good correspondences are hard to find

- Good news: Geometry constrains possible correspondences.
  - 4 DOF between x and x'; only 3 DOF in X.
  - Constraint is manifest in the **Fundamental matrix**
    \[ x'^T F x = 0. \]
  - F can be calculated either from camera matrices or a set of good correspondences.
Geometry of 2 views?

• What if we do not know $R, t$?

• Caveat:
  – My exposition uses different $R, t$
  – but more intuitive (IMHO)
  – I use $[R^T| -R^T t] = R^T [I| -t]$ camera matrices
  – Szeliski uses $[R| t]$
Epipolar Geometry

Where can \( p \) appear?

\[
M = [I|0]
\]

\[
M' = R^T[I|-t]
\]
Epipolar Lines

\[ M = \begin{bmatrix} I & 0 \end{bmatrix} \]

\[ M' = R^T \begin{bmatrix} I & -t \end{bmatrix} \]
Image of Camera Center

\[
M = [I|0]
\]

\[
M' = R^T[I|-t]
\]
Example:
Cameras Point at Each Other

**Top View**

**Epipolar Lines**

![Diagram of epipolar lines in top view](image)
Epipoles

- Epipoles inside the image: zoom-like setup.
Epipoles

- Epipoles in near-stereo config.
Epipolar Lines

\[ M = [I \mid 0] \]

\[ M' = R^T[I \mid -t] \]
Epipoles

• Camera Center $C'$ in first view:

$$e = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} t \end{bmatrix}_1 = t$$

• Origin $C$ in second view:

$$e' = \begin{bmatrix} R^T & -R^T t \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}_1 = -R^T t$$
Image of Camera Ray?

\[ M = [I|0] \]

\[ M' = R^T[I|-t] \]
Epipolar Lines

\[ M = [I|0] \]

\[ M' = R^T[I|-t] \]
Point at infinity

• Given $p'$, what is corresponding point at infinity $[x \ 0]$?

• Answer for any camera $M'=[A \ a]$:

\[ p' = [A \ a] \ast \begin{bmatrix} x \\ 0 \end{bmatrix} = A x \implies x = A^{-1} p' \]

• $A^{-1} = \text{Infinite homography}$

• In our case $M'=[R^T \ -R^T t]$: $x = Rp'$
Sidebar: Infinite Homographies

- Homography between
  - image plane
  - plane at infinity

- Navigation by the stars:
  - Image of stars = function of rotation R only!
  - Traveling on a sphere rotates viewer

https://www.usni.org/magazines/naval-history-magazine/2020/june/navigating-sextant
Epipolar Line Calculation

1) Point 1 = epipole e=t
2) Point 2 = point at infinity

\[
p_\infty = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} Rp' \\ 0 \end{bmatrix} = Rp'
\]

3) Epipolar line = join of points 1 and 2

\[
l = t \times Rp'
\]
Epipolar Lines

\[ p_{\infty} = Rp' \]

\[ l = t \times Rp' \]

\[ M = [I | 0] \]

\[ M' = R^T[I | -t] \]
Essential Matrix $E$

- $R, t = 6$ DOF
- However, scale! 
- $\Rightarrow 5$ DOF
Essential Matrix

- mapping from $p'$ to $l$

$$l = t \times R p' = \begin{bmatrix} t \end{bmatrix}_x R \cdot p' = E \cdot p'$$

- $E = 3 \times 3$ matrix

- Because $p$ is on $l$, we have

$$p^T E p' = 0$$
Epipolar lines

\[ e = t \]

\[ p_\infty = Rp' \]

\[ l = t \times Rp' \]
Epipolar Plane

\[ p, p' \]

\[ C, C' \]

\[ e = t \]

\[ M = [I|0] \]

\[ M' = R^T[I|-t] \]
Fundamental Matrix
Uncalibrated Case

$p_\infty = A^{-1}p'$

$l = [e] A^{-1} p'$

$M = [I|0]$  

$M' = [A|a]$
**Fundamental Matrix F**

- $3 \times 3 = 9$ DOF
- However, scale, rank 2!
- $\Rightarrow$ 7 DOF

\[
M = [I | 0]
\]
\[
M' = [A | a]
\]
Fundamental Matrix

• mapping from $p'$ to $l$

$$l = e \times A^{-1} p' = [e]_x A^{-1} \cdot p' = F \cdot p'$$

• $F = 3 \times 3$ matrix

• Because $p$ is on $l$, we have

$$p^T F p' = 0$$
Uncalibrated Case, Forsyth & Ponce Version

\[ \hat{p}^T \mathcal{E} \hat{p}' = 0 \]

\[ p = \mathcal{K} \hat{p} \]

\[ p' = \mathcal{K}' \hat{p}' \]

\[ p^T \mathcal{F} p' = 0 \quad \text{with} \quad \mathcal{F} = \mathcal{K}^{-T} \mathcal{E} \mathcal{K}'^{-1} \]

Fundamental Matrix
(Faugeras and Luong, 1992)
Properties of the Fundamental Matrix

• $Fp'$ is the epipolar line associated with $p'$.
• $F^Tp$ is the epipolar line associated with $p$.
• $Fe=0$ and $Fe'=0$.
• $F$ is singular.
Non-Linear Least-Squares Approach (Luong et al., 1993)

Minimize

$$\sum_{i=1}^{n} [d^2(p_i, \mathcal{F}p_i^l) + d^2(p_i^l, \mathcal{F}^T p_i)]$$

with respect to the coefficients of $\mathcal{F}$, using an appropriate rank-2 parameterization.

$d(p, l) = \text{point to line distance}$

$$= \frac{ax + by + cw}{\sqrt{a^2 + b^2}}$$

$d^2(p, l) = |ax + by + cw|^2/(a^2 + b^2)$
The Eight-Point Algorithm (Longuet-Higgins, 1981)

\[
\begin{pmatrix}
F_{11} & F_{12} & F_{13} \\
F_{21} & F_{22} & F_{23} \\
F_{31} & F_{32} & F_{33}
\end{pmatrix}
\begin{pmatrix}
u' \\
v'
\end{pmatrix} = 0
\]

Minimize:

\[
\sum_{i=1}^{n} (p_i^T F p'_i)^2
\]

under the constraint

\[
|F|^2 = 1.
\]
The Normalized Eight-Point Algorithm (Hartley, 1995)

- Center the image data at the origin, and scale it so the mean squared distance between the origin and the data points is 2 pixels:
  \[ q_i = T \, p_i \quad q'_i = T' \, p'_i. \]

- Use the eight-point algorithm to compute \( F \) from the points \( q_i \) and \( q'_i \).

- Enforce the rank-2 constraint.

- Output \( T^{-1} F T' \).