Contacts

Frank Dellaert
Center for Robotics and Intelligent Machines
Georgia Institute of Technology

March 11, 2020

This note introduces kinematic constraints and forces at contacts in a unified manner across $SO(2)$, $SO(3)$, $SE(2)$, and $SE(3)$ by using Lie group concepts. Further references (albeit *all* using slightly different notation from this note *and* each other) are the texts by Murray, Li, and Sastry [3], Matt Mason [2] and most recently, the excellent synthesis by Lynch and Park [1].

A word about notation: Suppose Ω and v are 3-vectors, and $\dot{\xi}$ a 6-vector, then I might write $\dot{\xi}$ in two different ways, depending on whether we mention it in the text, such as $\dot{\xi} = (x, y)$, or in a display formula, such as

$$\dot{\xi} = \begin{bmatrix} \Omega \\ v \end{bmatrix}$$

Mathematically, we think of $\dot{\xi}$ as a column vector, but the $(.)$ notation helps us by not having to always write transposes all over that clutter the notation, i.e., we have $\dot{\xi} = (\Omega, v) = \begin{bmatrix} \Omega^T & v^T \end{bmatrix}^T$.

1 Kinematic Constraints

At a contact with location p^S and contact normal $\bar{n}$, in some spatial frame S, we have a very simple constraint on the spatial velocity v^S at that point,

$$v^S \bar{n} \geq 0 \tag{1.1}$$

which is true in 2D and 3D. Below we show what that means for rotating bodies in 2D and 3D, and subsequently for rigidly moving bodies in 2D and 3D.

In each case, we proceed by expressing the spatial velocity v^S in terms of differential twist coordinates, and then deriving a constraint on those in terms of the contact parameters p^S and $\bar{n}$.

1.1 Planar Rotations aka SO(2)

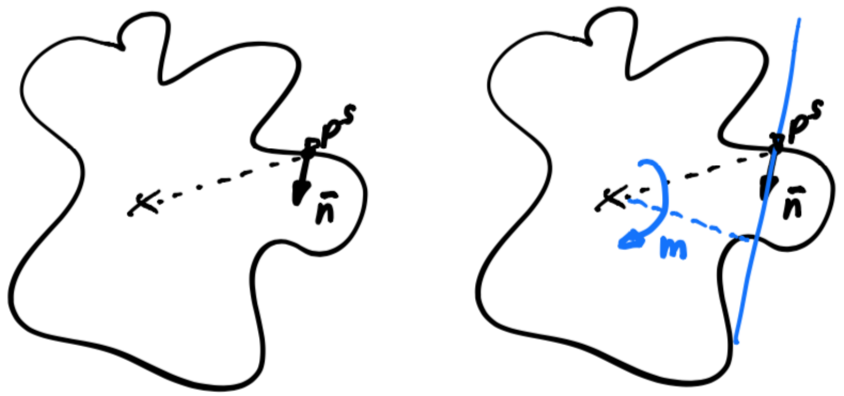

Figure 1.1: Left: contact constraining this object from rotating. Right: moment m of the contact normal is negative (clockwise) in the situation above.

Consider the situation in Figure 1.1, with contact blocking the rotation of the asteroid-like object from rotating. From the figure it is obvious that we will only be able to rotate clock-wise, away from the contact. Let us formalize this with math.

For 2D rotations, the point p^S describes a circular trajectory around the origin. The velocity v^S, in spatial coordinates, is given by

$$v^S = \hat{\omega} p^S = \omega p^{S\perp} \tag{1.2}$$

where ω is the 1-dimensional **angular velocity**, and $\hat{\omega} \in \mathfrak{so}(2)$ is given by

$$\hat{\omega} \triangleq \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}.$$

Given this, the kinematic constraint (1.1) becomes

$$\begin{aligned} v^S \bar{n} &\geq 0 \\ \hat{\omega} p^S \bar{n} &\geq 0 \\ \omega \left(p^{S\perp} \bar{n} \right) &\geq 0 \\ \omega m &\geq 0 \end{aligned}$$

Figure 1.2: Left: Contact constraining this object from rotating. Right: moment m of the contact normal is negative in this case.

where $m \triangleq p^{S\perp}\bar{n}$ is defined as the **moment** of the contact line through the contact point p^S. This moment quantity m is illustrated on the left of Figure 1.1.

The angular velocity ω is constrained to be either positive or negative, as illustrated in Figure 1.2.

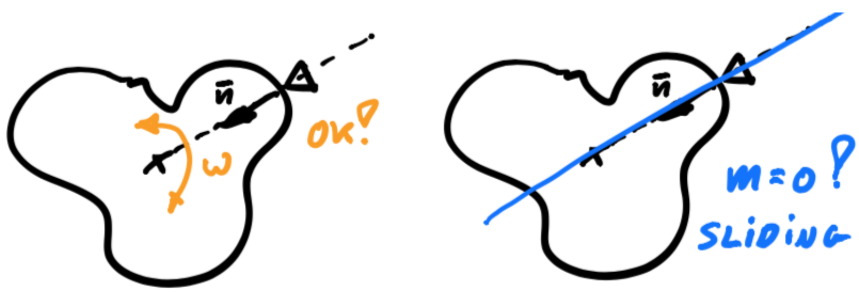

Figure 1.3: Left: Contact constraining this object from rotating. Right: moment m of the contact normal is negative in this case.

The exception occurs when the contact normal $\bar{n}$ is parallel to p^S, in which case we have a **sliding contact** if $\omega \neq 0$. This is illustrated in Figure 1.3.

1.2 Rotations in 3D aka SO(3)

For 3D rotations, the spatial velocity v^S of a point p^S is given by:

$$v^S = \hat{\Omega} p^S = \Omega \times p^S$$

where Ω is the **angular velocity vector** and $\hat{\Omega} \in \mathfrak{so}(3)$ is given by:

$$\widehat{\Omega} \triangleq \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.$$

Given this, the kinematic constraint (1.1) becomes

$$\begin{aligned} v^S \bar{n} &\geq 0 \\ \hat{\Omega} p^S \bar{n} &\geq 0 \\ \left(\Omega \times p^S\right) \bar{n} &\geq 0 \\ \Omega \left(p^S \times \bar{n}\right) &\geq 0 \\ \Omega m &\geq 0 \end{aligned}$$

where $m \triangleq p^S \times \bar{n}$ is defined as the **moment vector** of the contact line through the contact point p^S. The angular velocity vector Ω is constrained to be on the positive side of the plane with normal m, unless the contact normal $\bar{n}$ is parallel to p^S, in which case we have a sliding contact with non-zero Ω.

1.3 2D Rigid Transforms aka SE(2)

For 2D rigid transforms, the spatial velocity is given by

$$v^S = \hat{\dot{\xi}} p^S = \omega p^{S\perp} + v$$

where the 2D differential twist $\hat{\dot{\xi}} \in \mathfrak{se}(2)$ is given by

$$\hat{\dot{\xi}} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}.$$

Given this, the kinematic constraint (1.1) becomes

$$\begin{aligned} v^S \bar{n} &\geq 0 \\ \hat{\dot{\xi}} p^S \bar{n} &\geq 0 \\ \left(\omega p^{S\perp} + v\right) \bar{n} &\geq 0 \\ (m, \bar{n})^T (\omega, v) &\geq 0 \end{aligned}$$

where $(m, \bar{n}) = (p^{S\perp}\bar{n}, \bar{n})$ defines the contact line, through the equation $q^\perp \bar{n} = m$. The 3D twist coordinates $\xi = (\omega, v)$ are constrained to be on the positive side of the plane in 3D, with normal $(m, \bar{n})$. A sliding contact occurs when the twist coordinates ξ are *in* the plane and $v^S \neq 0$. A **rolling contact**, defined as having the spatial velocity v^S equal to zero for a non-zero twist, only occurs when the instantaneous rotation center $v^{S\perp}/\omega$ is equal to p^S:

$$\begin{aligned} \omega p^{S\perp} + v &= 0 \\ -\omega p^S + v^\perp &= 0 \\ v^\perp/\omega &= p^S \end{aligned}$$

Reuleaux' method is a great way to graph all possible IRCs in the plane: positive IRCs are possible to the left of the contact line, and negative to the right, both corresponding to **B**reaking contact. On the line, we have either **S**liding, or **R**olling (at the contact point). Sliding can further be subdivided in left or right sliding.

1.4 3D Rigid transforms aka SE(3)

For 3D rigid transforms, the spatial velocity is given by

$$v^S = \hat{\dot{\xi}} p^S = \Omega \times p^S + v$$

where the 3D differential twist $\hat{\dot{\xi}} \in \mathfrak{se}(3)$ is given by

$$\hat{\dot{\xi}} = \begin{bmatrix} \hat{\Omega} & v \\ 0 & 0 \end{bmatrix}.$$

Given this, the kinematic constraint (1.1) becomes

$$\begin{aligned} v^S \bar{n} &\geq 0 \\ \hat{\dot{\xi}} p^S \bar{n} &\geq 0 \\ \left(\Omega \times p^S + v\right) \bar{n} &\geq 0 \\ (m, \bar{n})^T (\Omega, v) &\geq 0 \end{aligned}$$

where $(m, \bar{n}) = (p^s \times \bar{n}, \bar{n})$ defines the contact line, through the equation $q \times \bar{n} = m$. The 6-dimensional vectors $(m, \bar{n})$ to represent lines in 3D are also known as the **Plücker coordinates** of a 3D line. The twist coordinates $\xi = (\Omega, v)$ are constrained to be on the positive side of the hyperplane in 6D with equation $(m, \bar{n})^T \xi = 0$. A sliding contact occurs when the twist coordinates ξ are *in* the plane and $v^S \neq 0$. A rolling contact occurs when $\xi \neq 0$ and

$$v^S = \Omega \times p^S + v = 0.$$

The intuition I can offer is this: in all 4 cases, some properties of the contact line determine the constraint (breaking contact, sliding, or disallowed). Looking in hindsight, the Plücker coordinates (note spelling!) determine the line by a moment and direction. When looking at rotation, only the *moment* of the line matters. When upgrading to Euclidean transforms, the exact direction of the line also matters.

2 Friction

The above was a kinematic account. To discuss frictional contacts and force closure we closely follow Murray et al [3], modulo some notation differences. In particular, we will arrive at the following concise result: the set F_B of possible wrenches applied to the body B is given by

$$F_B = \{Gf | f \in FC\}$$

where G is the $n \times m$ **grasp map**, and $f \in \mathbb{R}^m$ are set of possible forces, which lie inside a **friction cone** FC.

To see this, we will classify each of k contacts into different contact types, and for each contact c_i we model the contacts in their contact frame T_i^b as

$$\mathcal{F}_i = B_i f_i$$

where B_i is a **wrench basis**, and $f_i \in FC_i$ are the contact forces lying inside the contact's friction cone F_i. Below we assume that the contact frame is chosen such that the origin coincides with the point of contact, and the z-axis coincides with the contact normal. For spatial bodies, there are three different contact types we will consider [3]:

- Frictionless point contact:

$$B_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ FC_i = \{f_1 \geq 0\}$$

- Point contact with friction:

$$B_i = \begin{bmatrix} & & \\ & & \\ & & \\ 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \ F\left\{C_i = \sqrt{f_1^2 + f_2} \leq \mu f_3, \ f_3 \geq 0\right\}$$

- Soft-finger (note, $f_4$below has units of torque, not force):

$$B_i = \begin{bmatrix} & & & \\ & & & \\ & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix},\ FC_i = \left\{ \sqrt{f_1^2 + f_2} \leq \mu f_3,\ f_3 \geq 0, f_4 \leq \gamma f_3 \right\}$$

Expressing the wrench $\mathcal{F}_i$ applied by contact i in the body frame yields

$$\mathcal{F}_b = \left[Ad_{T_b^c}\right]^T \mathcal{F}_i = \left[Ad_{T_b^c}\right]^T B_i f_i = G_i f_i$$

Then the grasp map G and the generalized friction cone FC are given by

$$G = [G_1 \dots G_k],\ FC = FC_1 \cup \dots \cup FC_k$$

and the set F_B of possible wrenches applied by the contacts is given by

$$F_B = \{Gf | f \in FC\}$$

References

[1] Kevin M Lynch and Frank C Park. *Modern Robotics*. Cambridge University Press, 2017.

[2] Matthew T Mason. *Mechanics of robotic manipulation*. 2001.

[3] R.M. Murray, Z. Li, and S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.