Kinematics & Jacobians

8803 Mobile Manipulation

Topics

1. **Planar Geometry**
2. **Serial Link manipulators**
3. **Forward Kinematics**
4. **Describing Manipulators**
5. **Product of Exponentials**
6. **3D Geometry**

1. Planar Geometry

- SO(2)

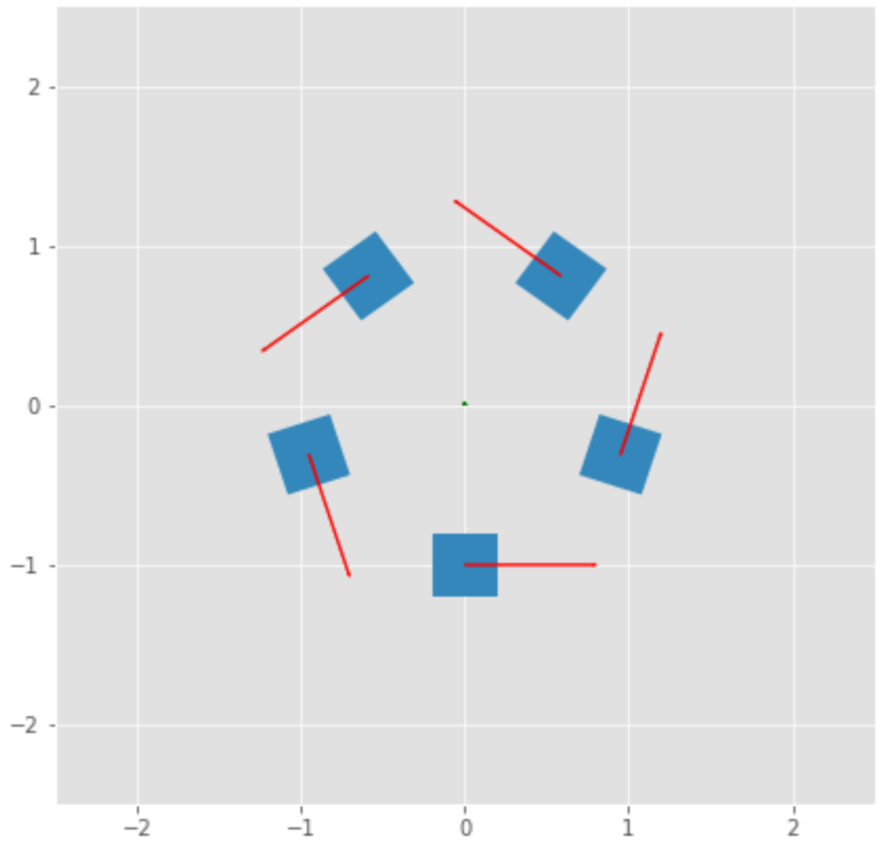
$$v^s(t) = \hat{\omega}(t)p^s(t) = \omega(t)\,[p^s(t)]^{\perp} \qquad \hat{\omega} \triangleq \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

- SE(2)

$$\begin{bmatrix} v^s(t) \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega}(t)p^s(t) + v(t) \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega}(t) & v(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p^s(t) \\ 1 \end{bmatrix}$$

$$\hat{\mathcal{V}} \triangleq \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

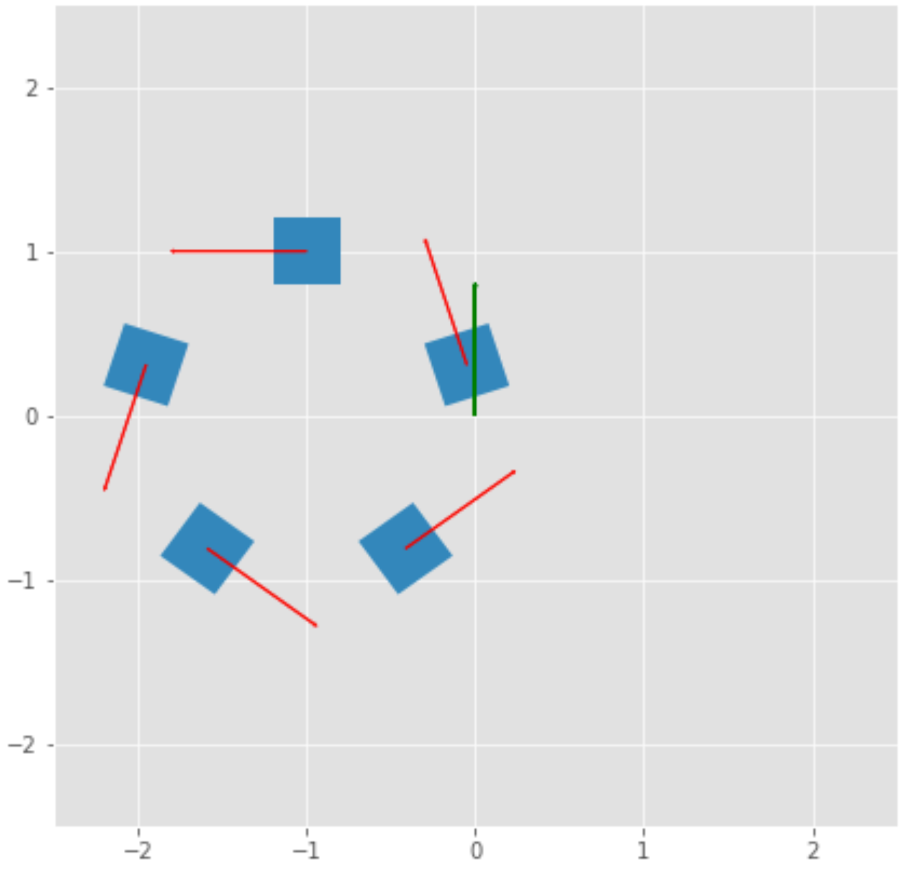
Body vs Spatial Twist



- Twist:
  - In body: [0, pi/4, pi/4]
  - Spatial: [0, 0, pi/4]

- IRC [-vy, vx]/w:
  - In body: [-1, 0]
  - Spatial: [0, 0]

Body vs Spatial Twist



- Twist:
  - In body: [0, pi/4, pi/4] (same !)
  - Spatial: [0, pi/4, pi/4]

- IRC [-vy, vx]/w:
  - In body: [-1, 0]
  - Spatial: [-1, 0]

Exponential Map for SO(2)

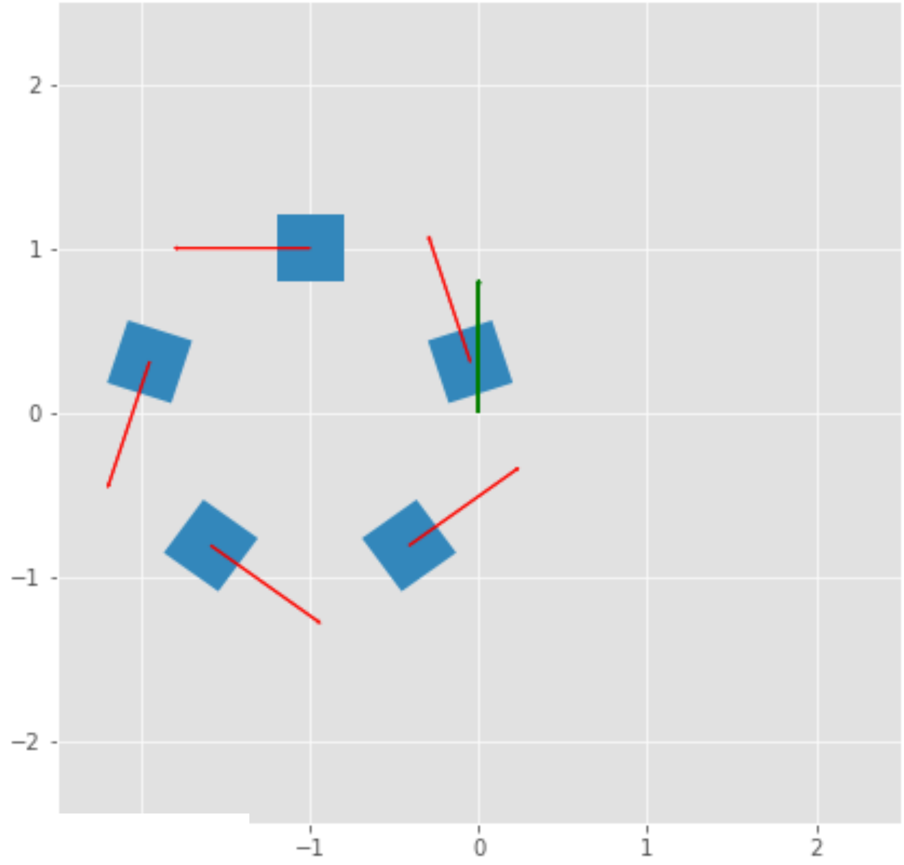
- ODE $$\frac{d}{dt}p^s(t) = \hat{\omega}p^s(t)$$

Solution: $$p^s(t) = \exp(\hat{\omega}t)p^s(0)$$

$$\exp(\hat{\theta}) \triangleq I + \hat{\theta} + \frac{\hat{\theta}^2}{2!} + \frac{\hat{\theta}^3}{3!} + \ldots$$

$$p^s(t) = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix} p^s(0)$$

Exponential Map for SE(2)



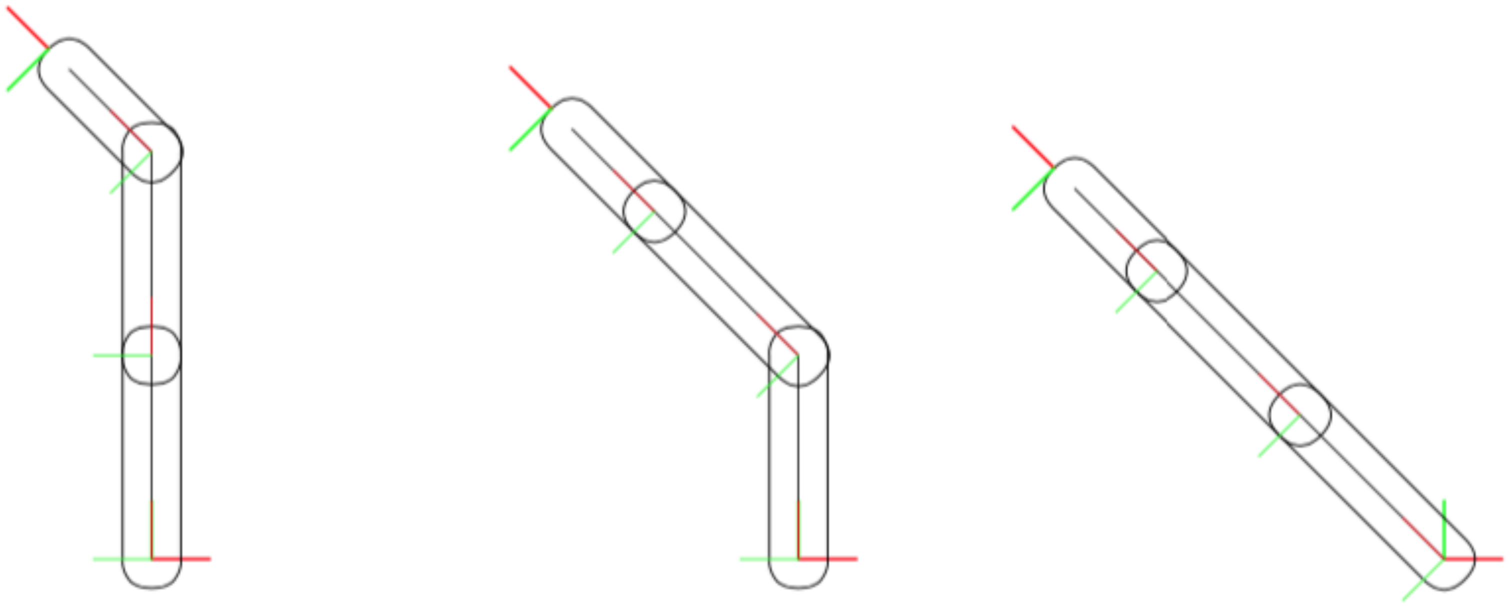
- SE(2)

$$\begin{bmatrix} p^s(t) \\ 1 \end{bmatrix} = \exp(\hat{\mathcal{V}}t) \begin{bmatrix} p^s(0) \\ 1 \end{bmatrix}$$

$$\exp\left(\hat{\mathcal{V}}t\right) = \begin{bmatrix} I & q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\omega t) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & -q \\ 0 & 1 \end{bmatrix}$$

$$q = v^{\perp}/\omega$$

2. Serial Manipulators



- RRR manipulator
- Joint j connects link i-1 and link i
- Base is link 0

3. Forward Kinematics

Given generalized joint coordinates $q \in Q$, we wish to determine the pose $T_t^s(q)$ of the tool frame T relative to the base frame S.

- Recursive, given joints 1..n:

$$T_t^s(q) = T_1^s(q_1) \ldots T_j^{j-1}(q_j) \ldots T_n^{n-1}(q_n) X_t^n.$$

4. Describing Serial Manipulators

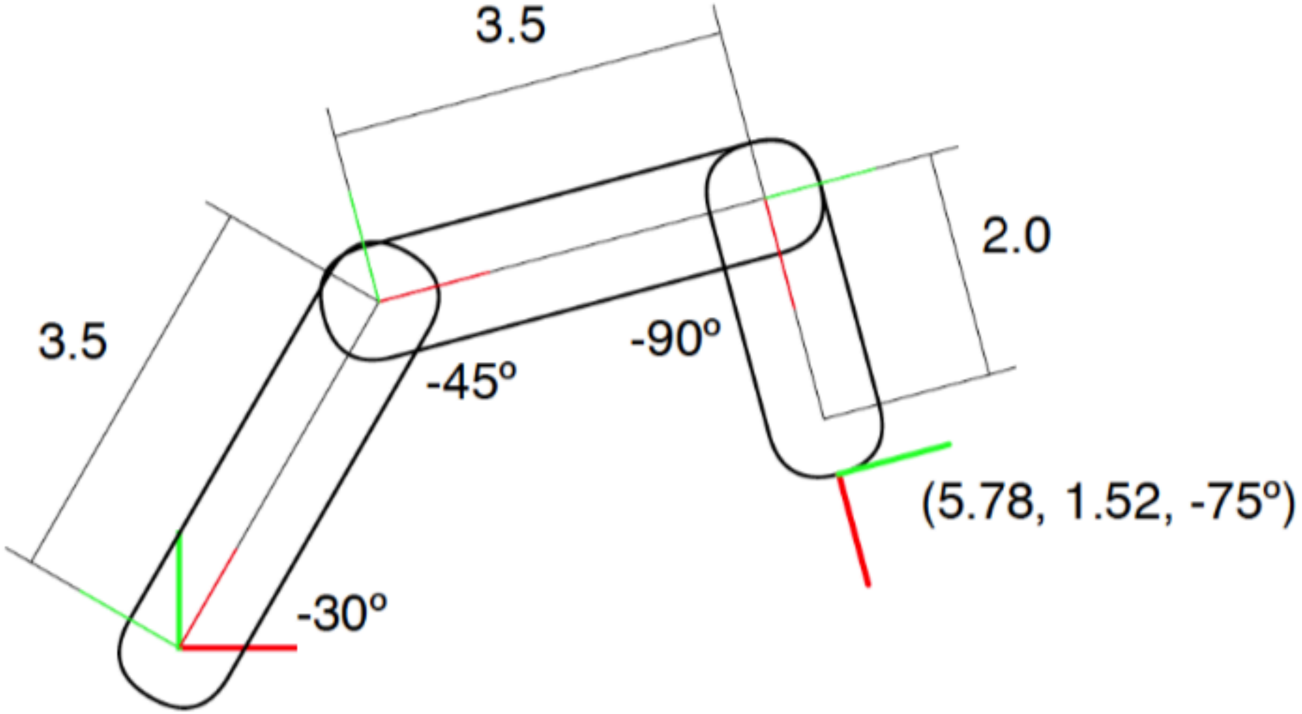
- Fixed offset X, joint transform Z

$$T_j^{j-1}(q_j) = X_j^{j-1} Z_j^j(q_j)$$

- FK becomes:

$$T_t^s(q) = X_1^s Z_1^1(q_1) \dots X_j^{j-1} Z_j^j(q_j) \dots X_n^{n-1} Z_n^n(q_n) X_t^n$$

Example RRR



$$T_t^s(\theta_1, \theta_2, \theta_3) = \{X_1^s Z_1^1(\theta_1)\}\{X_2^1 Z_2^2(\theta_j)\}\{X_3^2 Z_3^3(\theta_n)\} X_t^3$$

RRR

$$X_1^s = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } X_t^3 = \begin{bmatrix} 1 & 0 & 2.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X_2^1 = \begin{bmatrix} 1 & 0 & 3.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } X_3^2 = \begin{bmatrix} 1 & 0 & 3.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_t^s(q) = \begin{pmatrix} -\sin\beta & -\cos\beta & -3.5\sin\theta_1 - 3.5\sin\alpha - 2.5\sin\beta \\ \cos\beta & -\sin\beta & 3.5\cos\theta_1 + 3.5\cos\alpha + 3.5\cos\beta \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha = \theta_1 + \theta_2 \text{ and } \beta = \theta_1 + \theta_2 + \theta_3,$$

6. Product of Exponentials

- Joint at origin: real easy:

$$T_s^s(\theta) = \begin{bmatrix} R(\theta) & 0 \\ 0 & 1 \end{bmatrix}$$

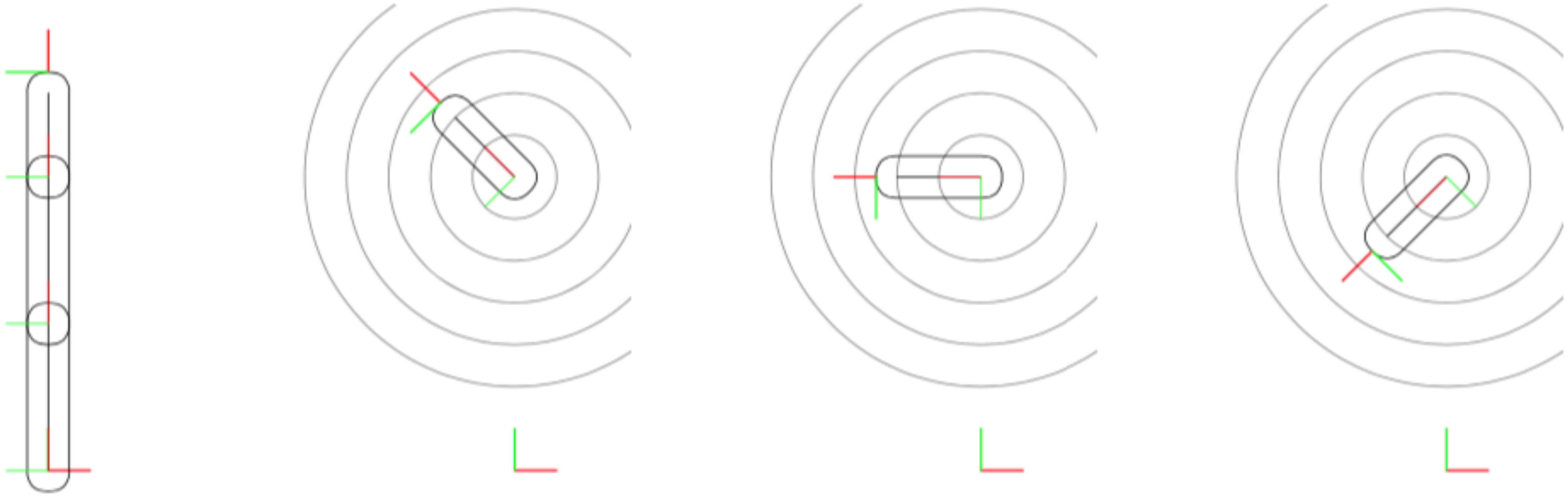
- Joint not at origin: conjugate!

$$T_s^s(\theta) = T_p^s \left\{ T_p^p(\theta) \right\} \left(T_p^s \right)^{-1} = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\theta) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & -p \\ 0 & 1 \end{bmatrix}$$

joint axis p is the **unit twist** $\mathcal{S} = (-p^{\perp}, 1) = (p_y, -p_x, 1)$

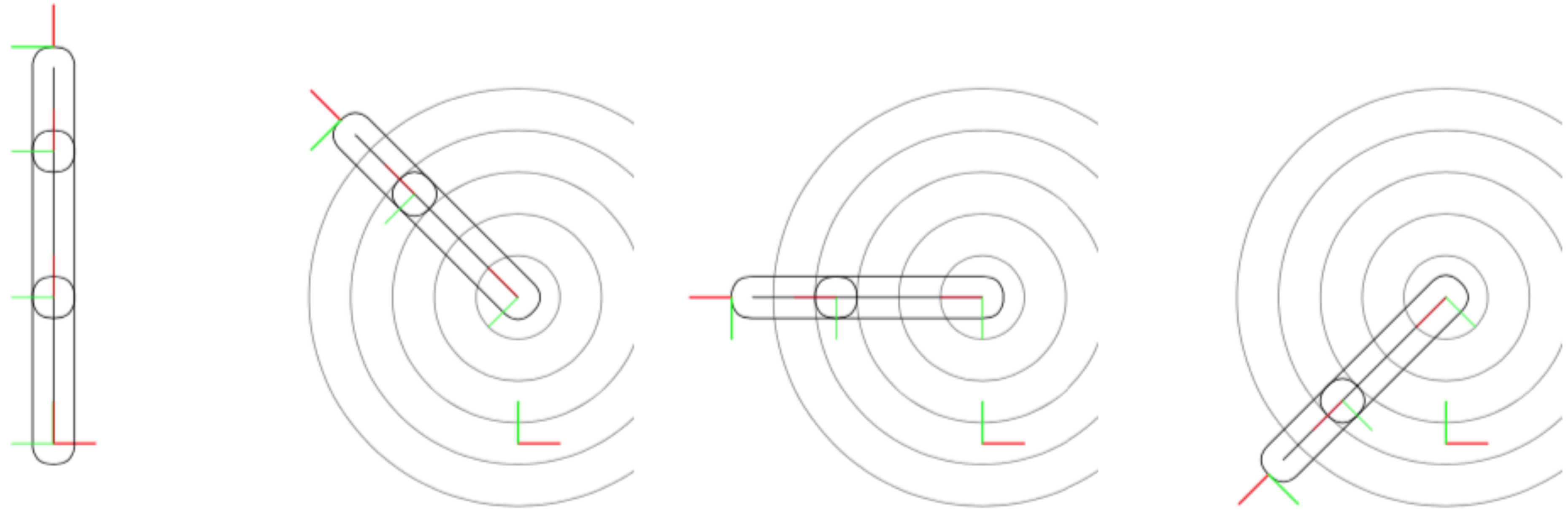
special case of the exponential map $\exp : \mathbb{R}^3 \rightarrow SE(2)$

Example



- Rotating around joint 3 and its effect on link 3

Example cont’d



- Rotating around joint 2, effect on link 2 *and* 3

General case: POE

In general, for any serial manipulator with n joints, we have the following product of exponentials expression for the forward kinematics,

$$T_t^s(q) = \exp\left(\hat{\mathcal{S}}_1\theta_1\right)\ldots\exp\left(\hat{\mathcal{S}}_j\theta_j\right)\ldots\exp\left(\hat{\mathcal{S}}_n\theta_n\right)T_t^s(0) \tag{6.5}$$

and, while the left-to-right order has to follow the manipulator structure, the formula above does *not* depend on the order in which the actual joints are actuated.

7. 3D Geometry

1. As a matrix group: $SE(3) \subset GL(4)$:

$$T_b^s = \begin{bmatrix} R_b^s & t_b^s \\ 0 & 1 \end{bmatrix}$$

2. Spatial velocity:

$$v^s(t) = \Omega(t) \times [p^s(t) - q^s(t)] + \lambda\Omega(t)$$

3. Lie algebra isomorphic to $\mathbb{R}^6$, the space of 3D differential twist coordinates $\mathcal{V} \triangleq (\Omega, v)$. Hat operator given by:

$$\hat{\mathcal{V}} \in \mathfrak{se}(3) = \begin{bmatrix} \hat{\Omega} & v \\ 0 & 0 \end{bmatrix}$$

4. The exponential map, with $q = (\Omega \times v)/\omega^2$:

$$\exp\left(\hat{\mathcal{V}}t\right) = \begin{bmatrix} I & q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{\hat{\Omega}t} & (\lambda t)\Omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & -q \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\Omega}t} & \left[I - e^{\hat{\Omega}t}\right] Nv/\omega + nn^T vt \\ 0 & 1 \end{bmatrix}$$

Spatial Manipulators

- Just use 3D twists

As defined in Lynch & Park, a unit twist is either a pure velocity or a screw:

- if $\omega = 0$ then $\mathcal{S} = (0, v/\|v\|)$, and corresponding twist $\mathcal{V} = \mathcal{S}v = (0, v)$;
- if $\omega \neq 0$ then $\mathcal{S} = (\Omega/\omega, v/\omega) = (n, v/\omega)$, and corresponding twist $\mathcal{V} = \mathcal{S}\omega = (\Omega, v)$.