Transforming Twists and Wrenches with the Adjoint Map in 3D

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1 3D Rigid Transformations

The Lie group $SE(3)$ is a subgroup of the general linear group $GL(4)$ of 4×4 invertible matrices of the form

$$T \triangleq \left[\begin{array}{cc} R & t \\ 0 & 1 \end{array} \right]$$

where $R \in SO(3)$ is a rotation matrix and $t \in \mathbb{R}^3$ is a translation vector.

Its Lie algebra $\mathfrak{se}(3)$ is the vector space of 4×4 twists $\hat{\mathcal{V}}$ parameterized by the *twist coordinates* $\mathcal{V} \in \mathbb{R}^6$, with the mapping [1]

$$\mathcal{V} \triangleq \left[\begin{array}{c} \Omega \\ v \end{array} \right] \rightarrow \hat{\mathcal{V}} \in \mathfrak{se}(3) = \left[\begin{array}{cc} [\Omega]_\times & v \\ 0 & 0 \end{array} \right]$$

where $[.]_\times$ denotes the skew-symmetric matrix of a 3-vector, i.e., the $SO(3)$ operator. Note we follow Lynch & Park's convention and reserve the first three components for rotation, and the last three for translation.

2 A Body moving in Space

For a body undergoing a constant twist $\mathcal{V}^s$, all the points p^s on that body are moving according to, in homogeneous coordinates:

$$v^s = \widehat{\mathcal{V}^s} p^s$$

hence in non-homogeneous coordinates:

$$v^s = \Omega \times p^s + v$$

Which coordinates we use should be obvious from context.

3 The Adjoint Map

Suppose we want to express the twist in the body frame B ? We start from

$$v^s = \widehat{\mathcal{V}^s} p^s$$

but we know that at any given moment $v^s = T^s_b v^b$, and $p^s = T^s_b p^b$. Substituting in, we have

$$T^s_b v^b = \widehat{\mathcal{V}^s} T^s_b p^b$$

$$v^b = [T_b^s]^{-1}\,\widehat{\mathcal{V}^s} T_b^s p^b$$

or, in other words

$$\widehat{\mathcal{V}^b} = [T_b^s]^{-1}\,\widehat{\mathcal{V}^s} T_b^s = T_s^b \widehat{\mathcal{V}^s} T_b^s = T_s^b \widehat{\mathcal{V}^s} \left[T_s^b\right]^{-1}$$

The last form above we call **Adjoint Map**, with symbol and dropping the indices for now:

$$\begin{aligned}
Ad_T \widehat{\mathcal{V}} &= T \widehat{\mathcal{V}} T^{-1} \\
&= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [\Omega]_\times & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} [R\Omega]_\times & -[R\Omega]_\times t + Rv \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} [R\Omega]_\times & t \times R\Omega + Rv \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

We recognize the last line as the hat of another twist, with $\Omega' = R\Omega$ and $v' = t \times R\Omega + Rv$. Hence, we can express the body twist $\mathcal{V}^b$ in terms of the spatial twist $\mathcal{V}^s$ by the Adjoint map as follows:

$$\mathcal{V}^b = \begin{bmatrix} \Omega^b \\ v^b \end{bmatrix} = \begin{bmatrix} R_s^b & 0 \\ [t_s^b]_\times R_s^b & R_s^b \end{bmatrix} \begin{bmatrix} \Omega^s \\ v^s \end{bmatrix}$$

We call this 6×6 matrix the Adjoint map matrix and write

$$\mathcal{V}^b = \left[Ad_{T_s^b}\right] \mathcal{V}^s$$

and of course vice versa:

$$\mathcal{V}^s = \left[Ad_{T_b^s}\right] \mathcal{V}^b$$

Note that there is a subtle but important distinction: Ad_T operates on 4×4 Lie algebra elements $\widehat{\mathcal{V}}$, and $[Ad_T]$ multiplies 6×1 twist coordinates $\mathcal{V}$.

4 Wrenches

Wrenches $\mathcal{F}$ are dual to twists, and so they transform inversely. The dot product of a twist and a wrench is power, and of course the power has to be the same in both spatial and body frame. Hence we have

$$\mathcal{F}^{sT} \mathcal{V}^s = \mathcal{F}^{bT} \mathcal{V}^b$$

and if we substitute in $\mathcal{V}^b = \left[Ad_{T_s^b}\right] \mathcal{V}^s$, we obtain

$$\mathcal{F}^{sT}\mathcal{V}^s = \mathcal{F}^{bT} \left[Ad_{T_s^b}\right] \mathcal{V}^s$$

Since this must hold for all possible twists $\mathcal{V}^s$, we have

$$\mathcal{F}^{sT} = \mathcal{F}^{bT} \left[Ad_{T_s^b}\right]$$

or (transposing everything)

$$\mathcal{F}^s = \left[Ad_{T_s^b}\right]^T \mathcal{F}^b$$

Switching the indices yields the other direction:

$$\mathcal{F}^b = \left[Ad_{T_b^s}\right]^T \mathcal{F}^s$$

References

[1] R.M. Murray, Z. Li, and S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.