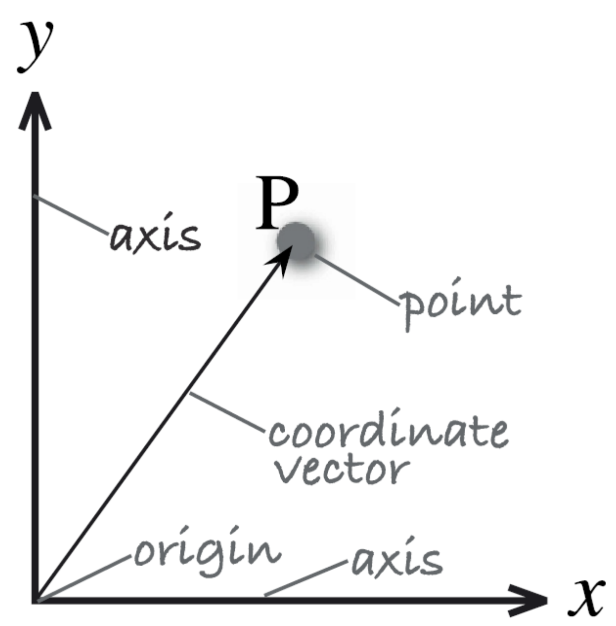
CS 3630

Pose in the Plane

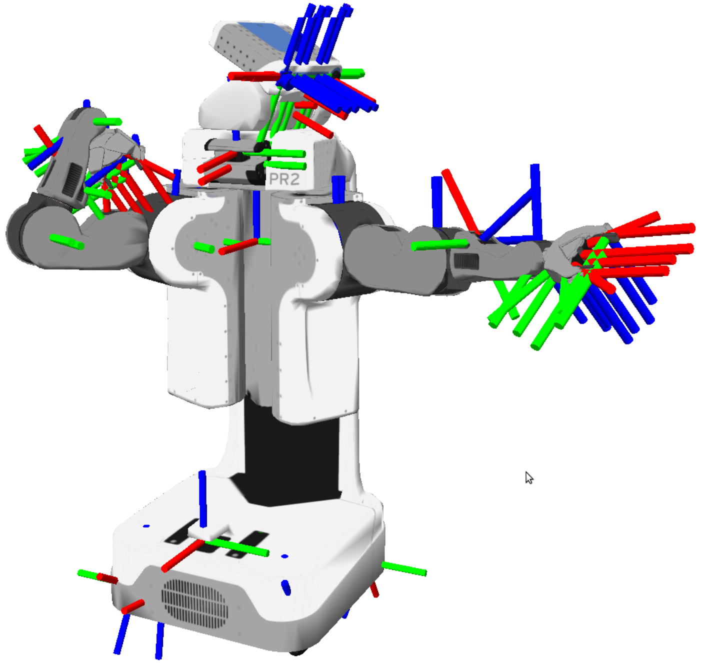
Reference Frames

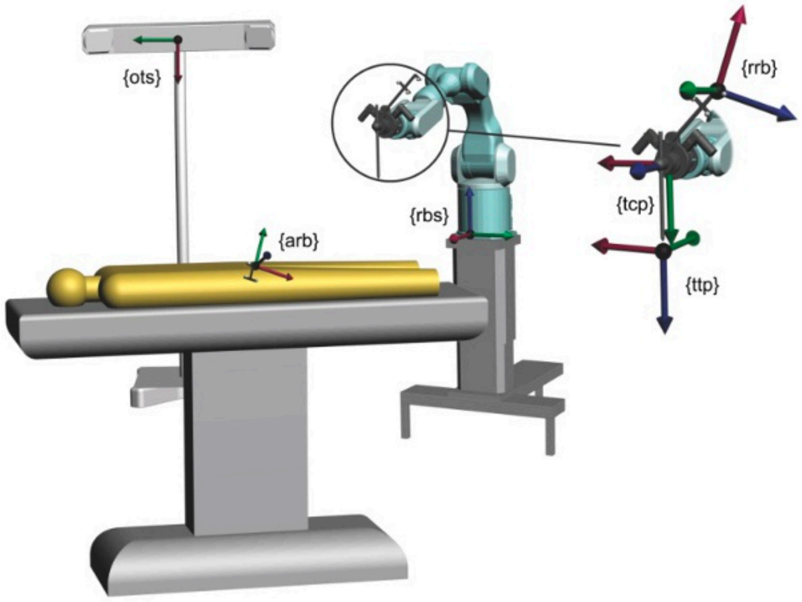
- Robotics is all about management of reference frames
  - **Perception** is about estimation of reference frames
  - **Planning** is how to move reference frames
  - **Control** is the implementation of trajectories for reference frames
- The relation between references frames is essential to a successful system

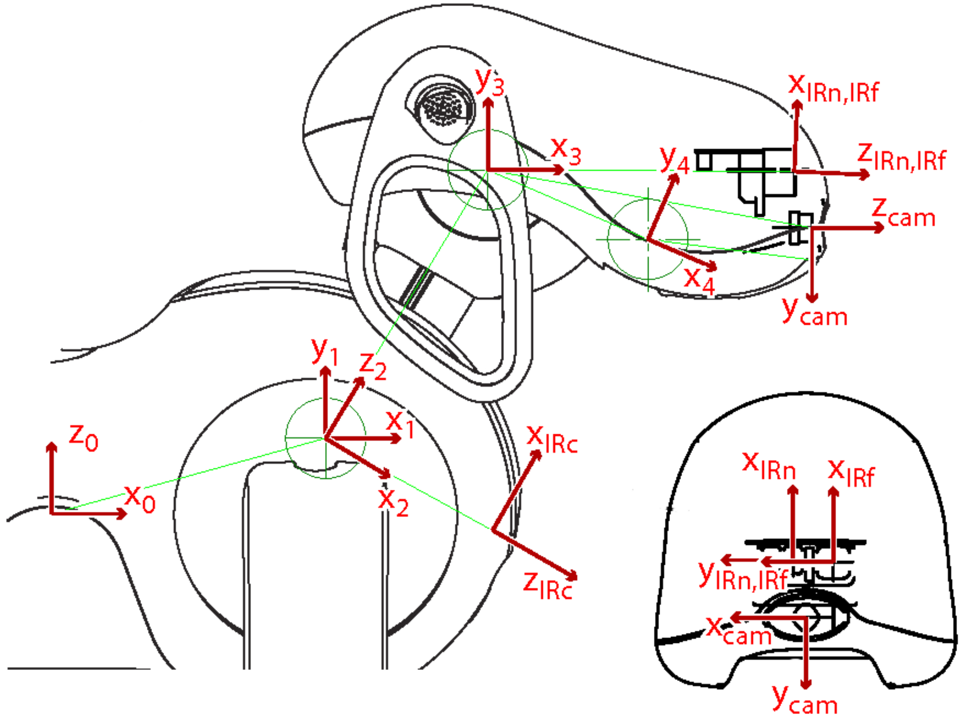


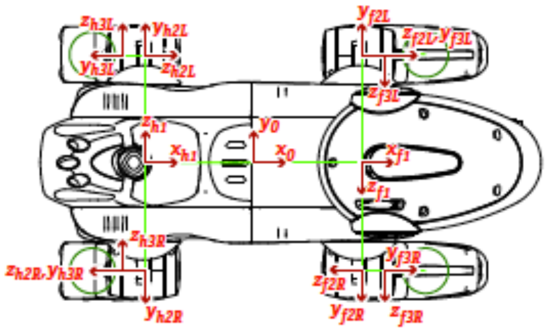
Examples of the types of reference frames we're talking about

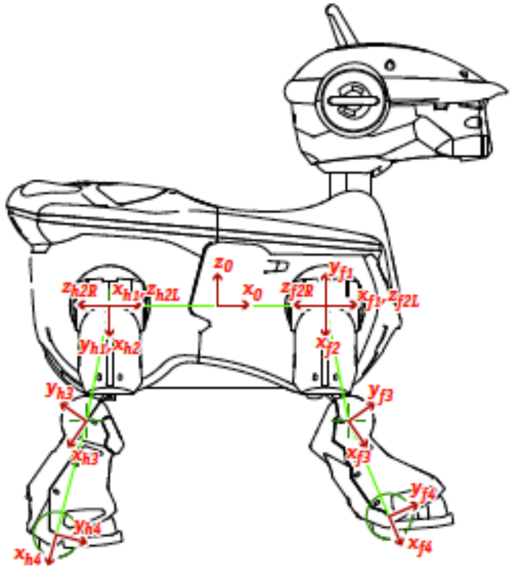
We rigidly attach coordinate frames to objects of interest. To specify the position and orientation of the object, we merely specify the position and orientation of the attached coordinate frame.

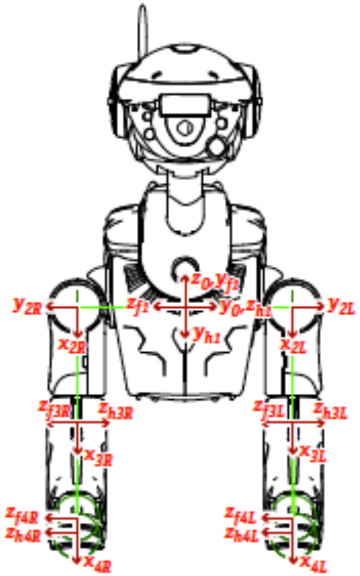










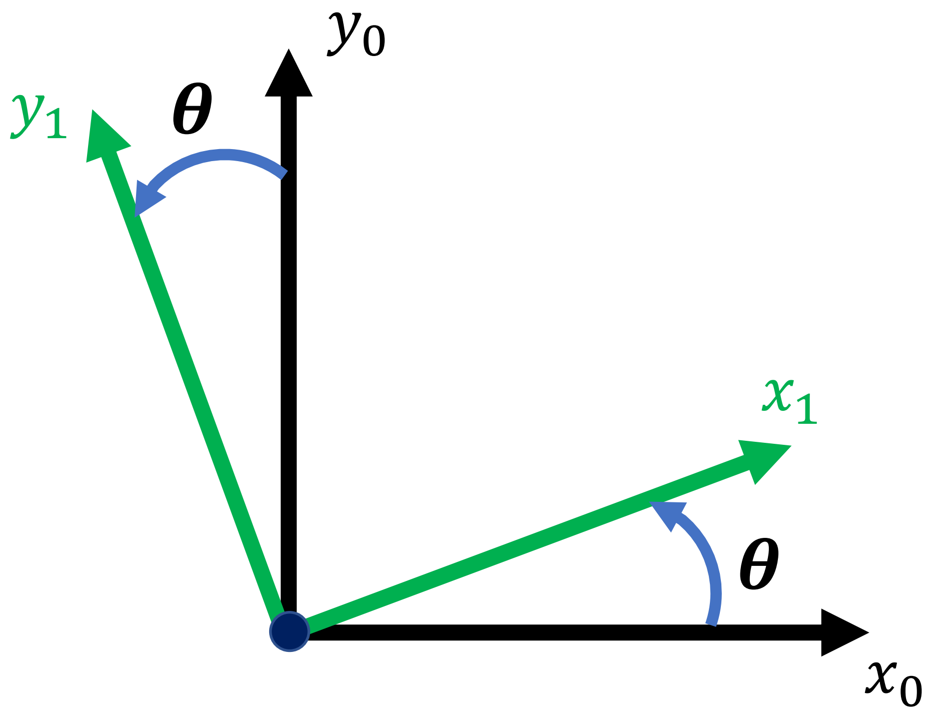


- The relationship between frames is often very simple to define, as in the case when two frames are related by the motion of a single joint/motor.
- For example the upper and lower leg of the dog robot are related by a single motor at the knee.

Today – we consider only the case of 2D reference frames, corresponding to mobile robots moving in the plane.

Specifying Orientation in the Plane

Given two coordinate frames with a common origin, how should we describe the orientation of Frame 1 w.r.t. Frame 0?



The obvious choice is to merely use the angle θ. This isn't a great idea for two reasons:

- We have problems at $\theta = 2\,\pi - \epsilon$. For ϵ near 0, we approach a discontinuity: for small change in ϵ, we can have a large change in θ.
- This approach does not generalize to rotations in three dimensions (and not all robots live in the plane).

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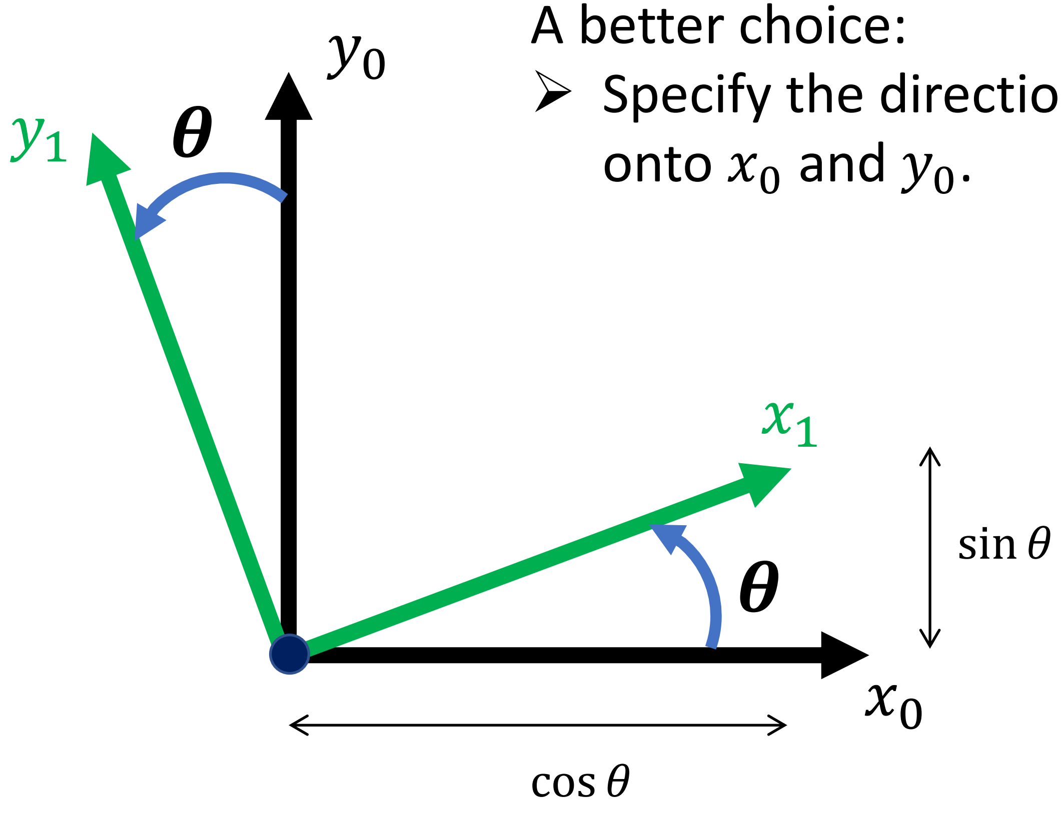


A better choice:

- Specify the directions of x_1 and y_1 with respect to Frame 0 by projecting onto x_0 and y_0.

$$x_1^0 = \begin{bmatrix} x_1 \cdot x_0 \\ x_1 \cdot y_0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Notation: x_1^0 denotes the x-axis of Frame 1, specified w.r.t Frame 0.

$$y_1^0 = \begin{bmatrix} y_1 \cdot x_0 \\ y_1 \cdot y_0 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

We obtain y_1^0 in the same way.

Rotation Matrices (rotation in the plane)

We combine these two vectors to obtain a ***rotation matrix***: $R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

All rotation matrices have certain properties:

1. The two columns are each unit vectors.
2. The two columns are orthogonal, i.e., $c_1 \cdot c_2 = 0$.
3. $\det R = +1$

For such matrices $R^{-1} = R^T$

- The first two properties imply that the matrix R is ***orthogonal***.
- The third property implies that the matrix is ***special***! (After all, there are plenty of orthogonal matrices whose determinant is -1, not at all special.)

The collection of 2×2 rotation matrices is called the ***Special Orthogonal Group of order 2***, or, more commonly $\boldsymbol{SO(2)}$.

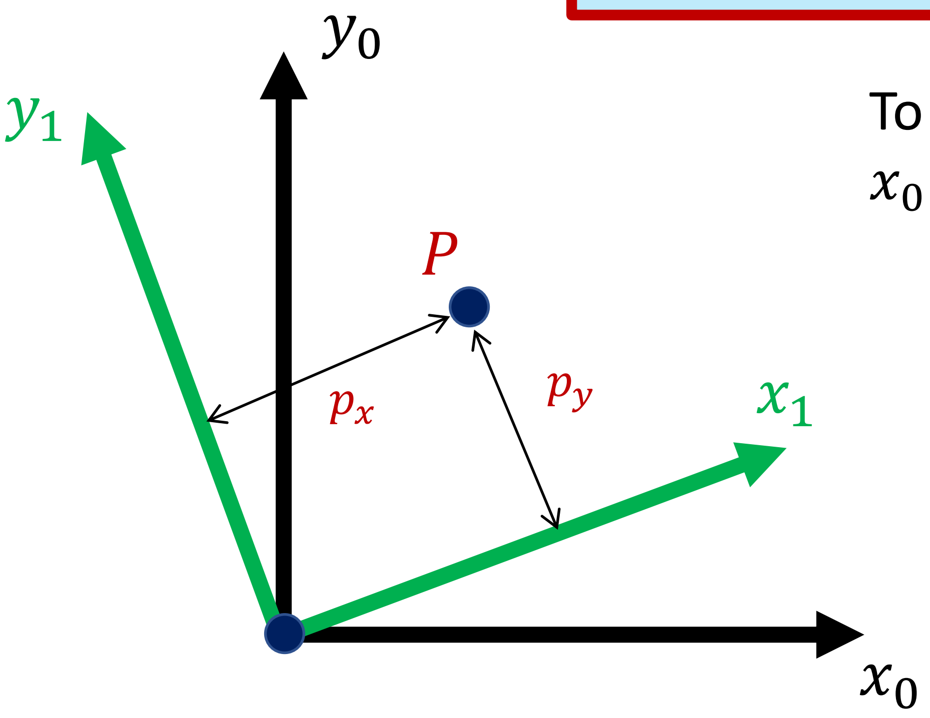
This concept generalizes to $\boldsymbol{SO(n)}$ for $n \times n$ rotation matrices.

Coordinate Transformations (rotation only)

Suppose a point P is rigidly attached to coordinate Frame 1, with coordinates given by $P^1 = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$.

We can express the location of the point P in terms of its coordinates

$$P = p_x x_1 + p_y y_1$$



To obtain the coordinates of P w.r.t. Frame 0, we project P onto the x_0 and y_0 axes:

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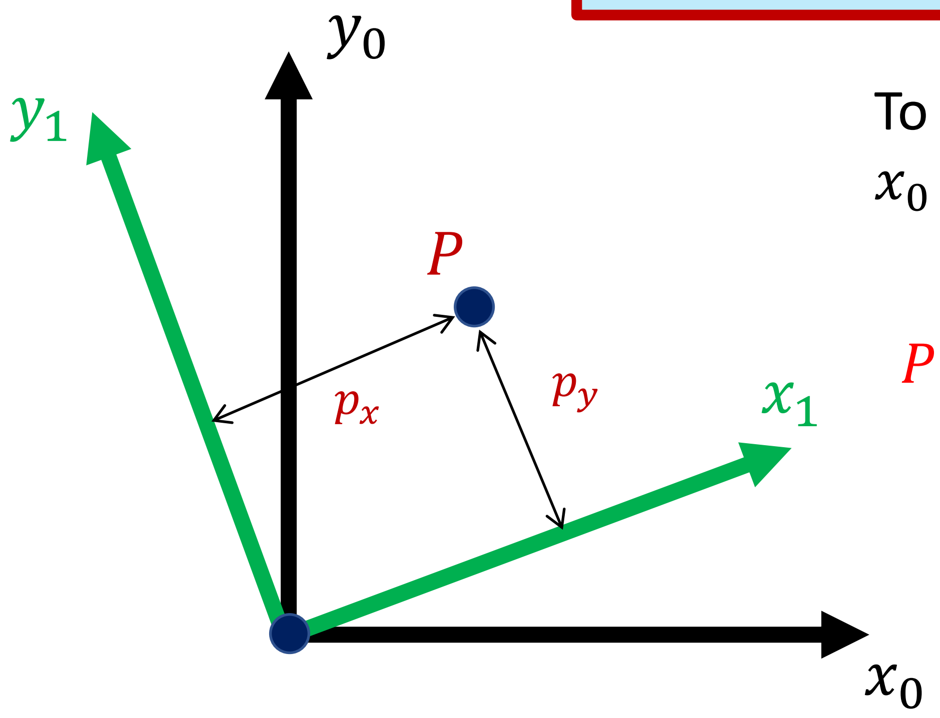

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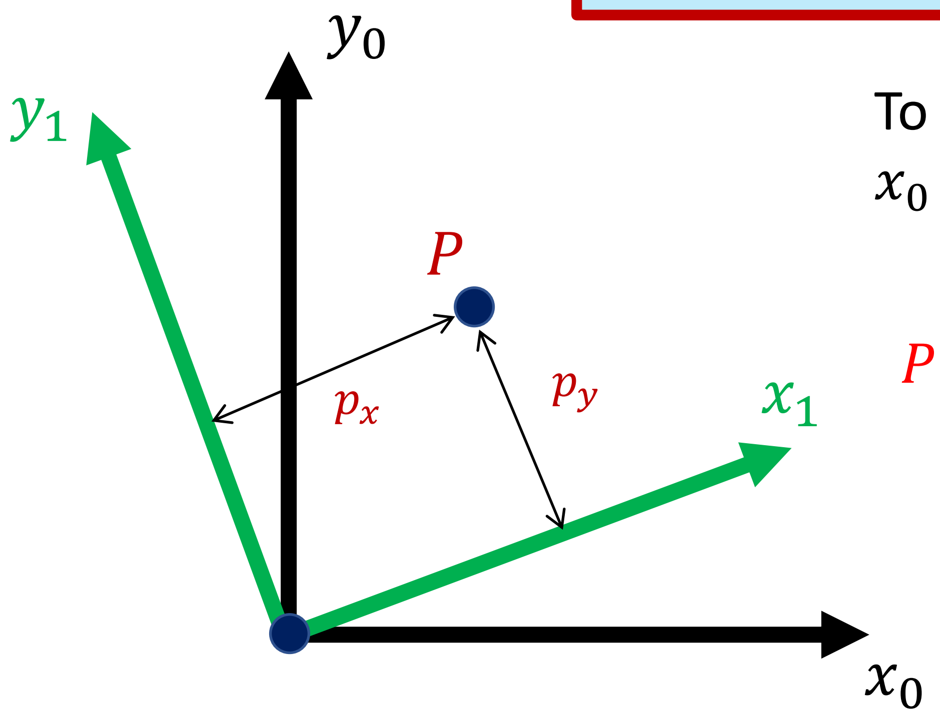

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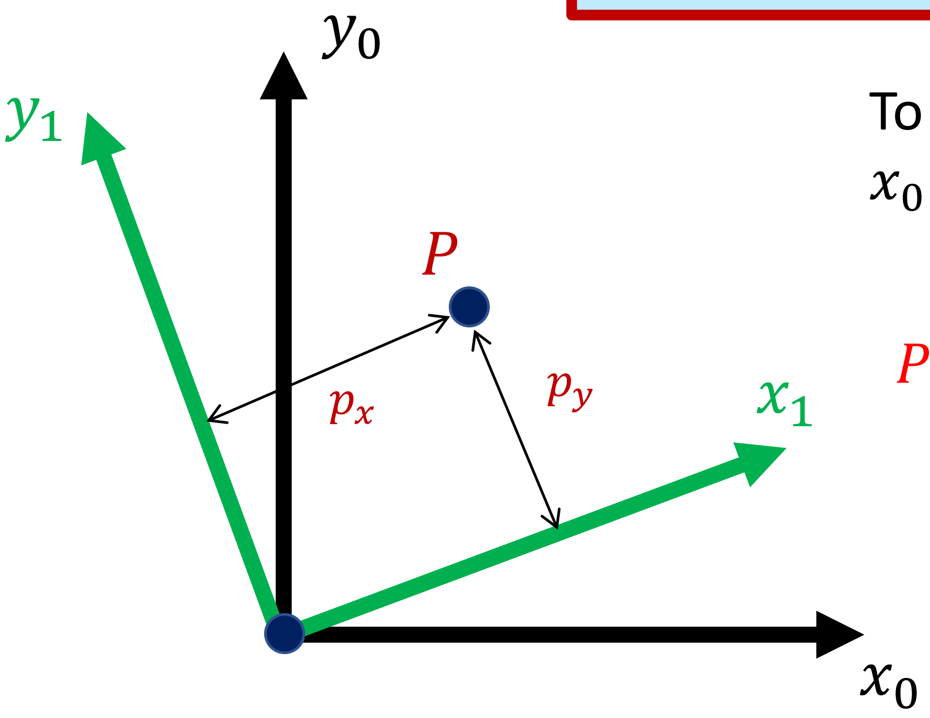

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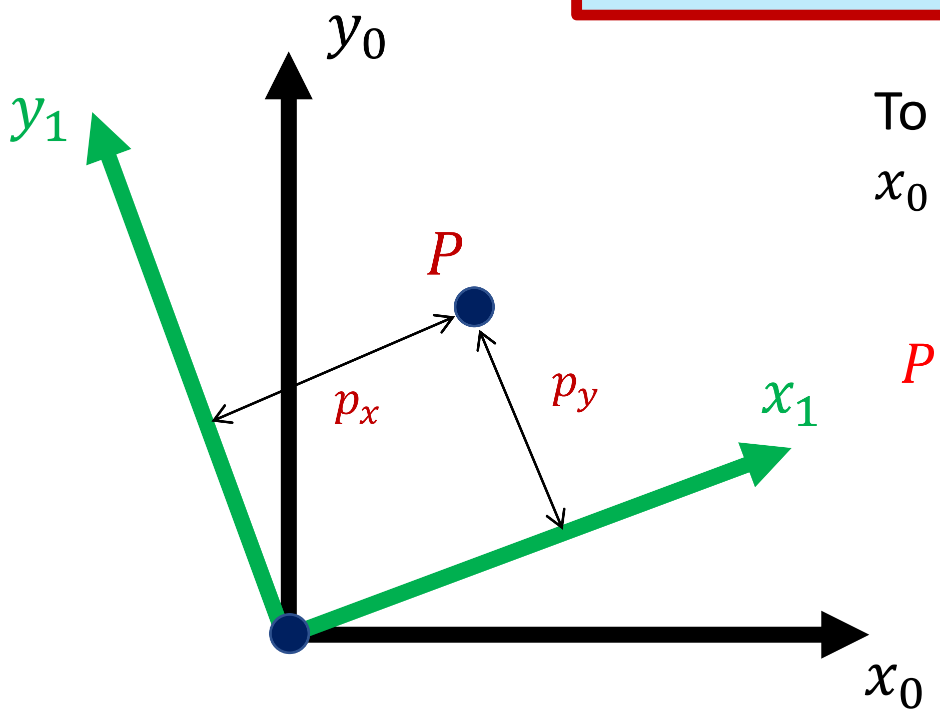

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$$= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

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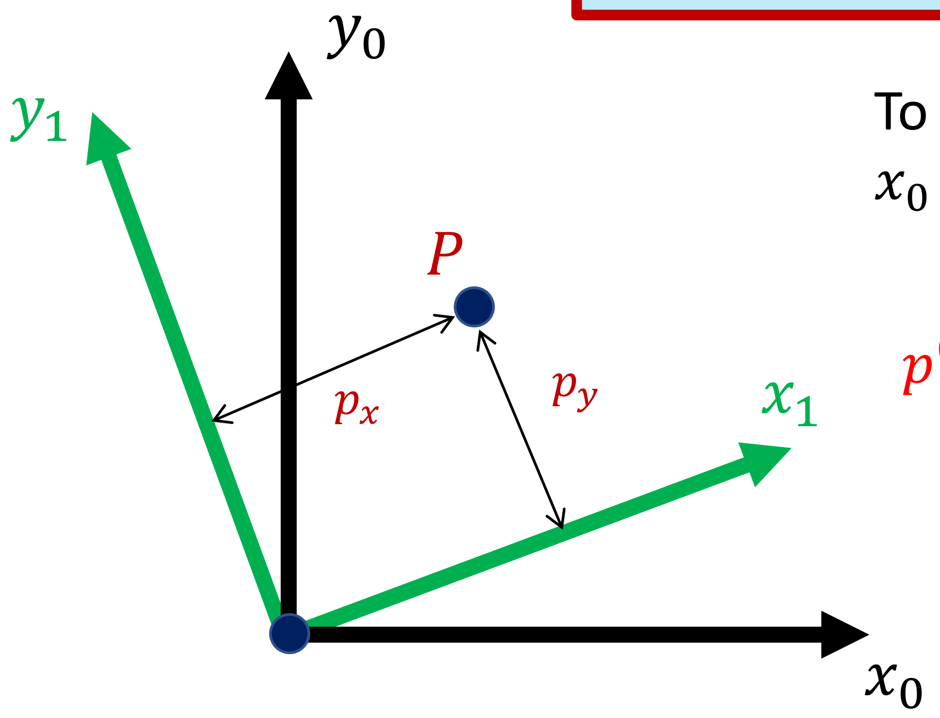

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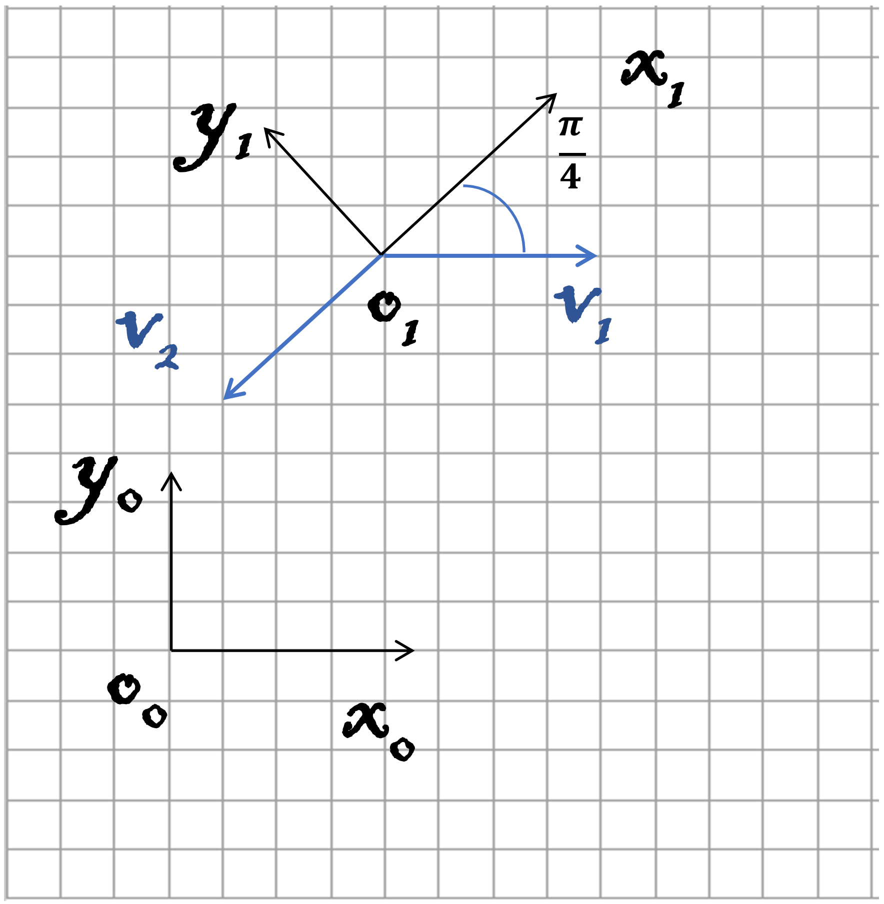
$$= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \boldsymbol{R_1^0}\, \boldsymbol{P^1}$$

$$\boldsymbol{P^0 = R_1^0\, P^1}$$

Lets practice…



- Two coordinate frames: o_0 and o_1
- Two free vectors: v_1 and v_2

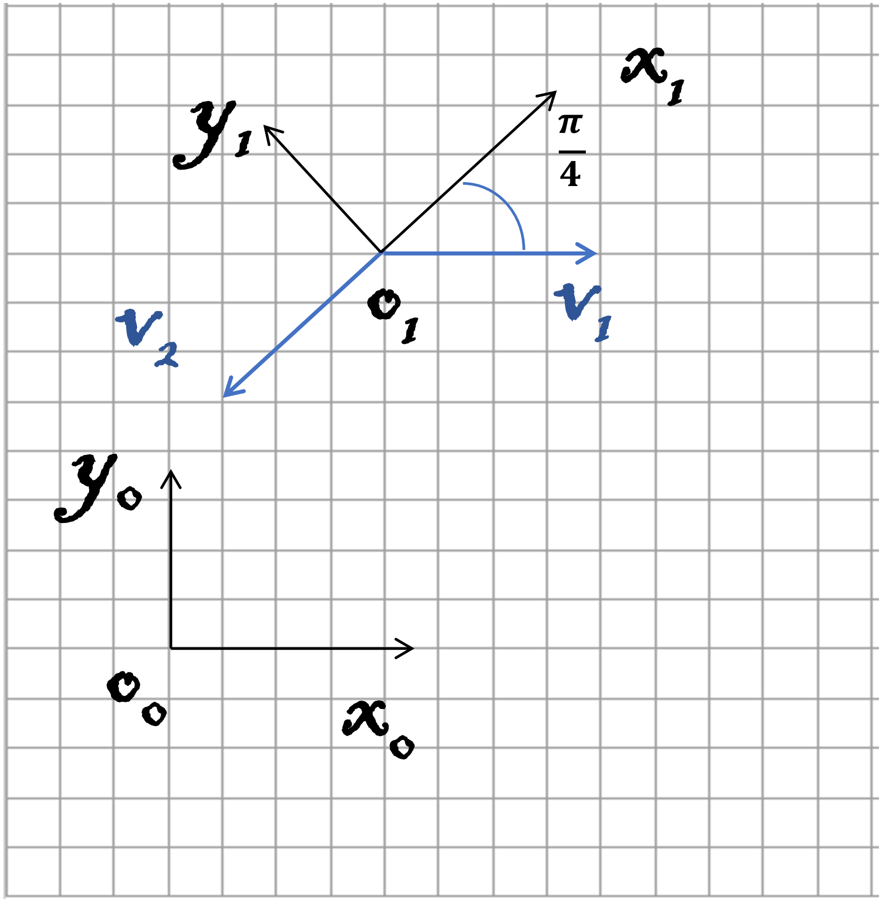
$$v_1^0 = [\quad] \qquad v_1^1 = [\quad]$$

$$v_2^0 = [\quad] \qquad v_2^1 = [\quad]$$

Recall: $\cos\frac{\pi}{4} = 0.5\sqrt{2}$, $\quad \sin\frac{\pi}{4} = 0.5\sqrt{2}$

Lets practice…

Note: $\|v_1\| = 4,\ \|v_2\| = 3\sqrt{2},$



- Two coordinate frames: o_0 and o_1
- Two free vectors: v_1 and v_2

$$v_1^0 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \qquad v_1^1 = \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$$

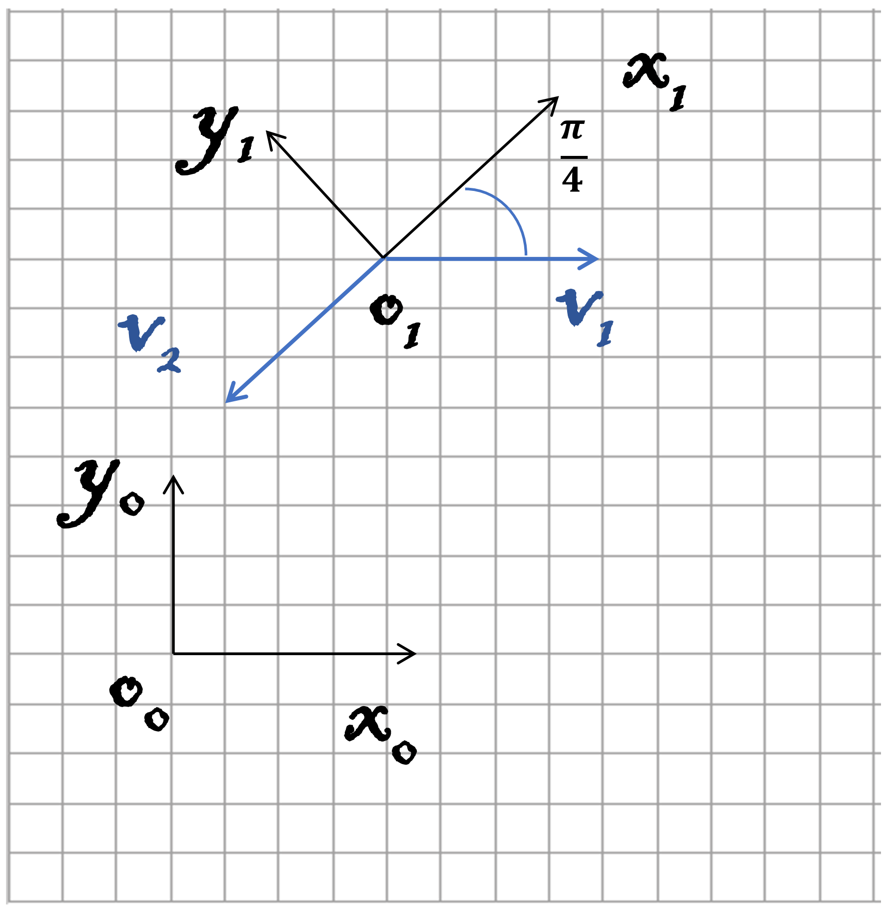
$$v_2^0 = \begin{bmatrix} -3 \\ -3 \end{bmatrix} \qquad v_2^1 = \begin{bmatrix} -3\sqrt{2} \\ 0 \end{bmatrix}$$

Recall: $\cos\frac{\pi}{4} = 0.5\sqrt{2}, \qquad \sin\frac{\pi}{4} = 0.5\sqrt{2}$

We can do this with simple geometry.

$$v_1^0 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad v_2^0 = \begin{bmatrix} -3 \\ -3 \end{bmatrix} \quad v_1^1 = \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix} \quad v_2^1 = \begin{bmatrix} -3\sqrt{2} \\ 0 \end{bmatrix}$$

More Practice…



$$R_1^0 = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} \\ 0.5\sqrt{2} & 0.5\sqrt{2} \end{bmatrix} \qquad R_0^1 = \begin{bmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} \\ -0.5\sqrt{2} & 0.5\sqrt{2} \end{bmatrix}$$

$$v_1^0 = R_1^0 v_1^1 = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} \\ 0.5\sqrt{2} & 0.5\sqrt{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$v_2^0 = R_1^0 v_2^1 = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} \\ 0.5\sqrt{2} & 0.5\sqrt{2} \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

$$v_1^1 = R_0^1 v_1^0 = \begin{bmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} \\ -0.5\sqrt{2} & 0.5\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$$

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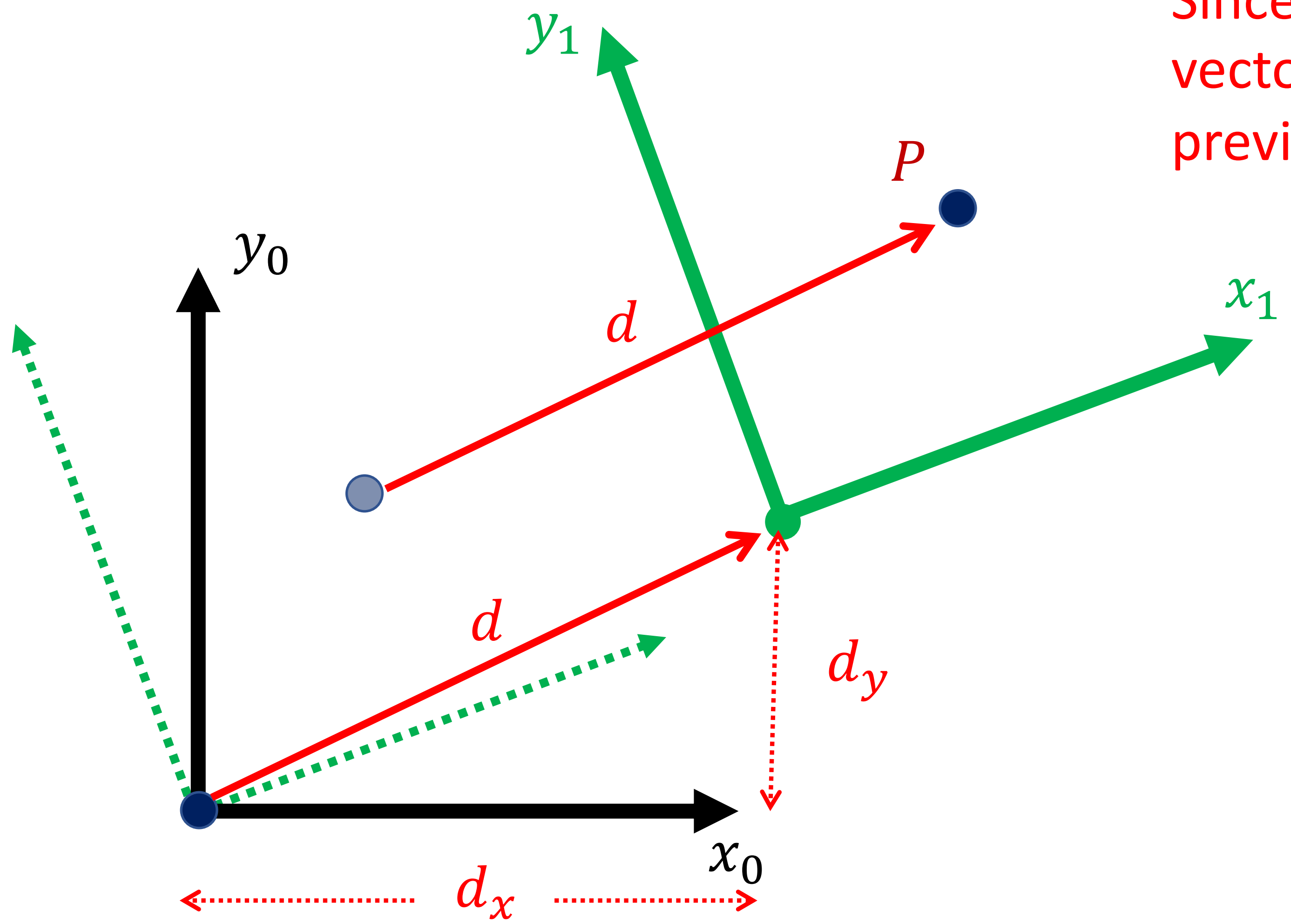
OR, we can use coordinate transformations!

Specifying Pose in the Plane

Suppose we now translate Frame 1 (*no new rotatation*).
What are the coordinates of P w.r.t. Frame 0?



Since we merely translated P by a fixed vector d, simply add the offset to our previous result!

$$\boldsymbol{P^0} = \boldsymbol{R_1^0}\,\boldsymbol{P^1} + \boldsymbol{d^0}$$

$$\boldsymbol{d^0} = \begin{bmatrix} \boldsymbol{d_x} \\ \boldsymbol{d_y} \end{bmatrix}$$

Homogeneous Transformations

We can simplify the equation for coordinate transformations by augmenting the vectors and matrices with an extra row:

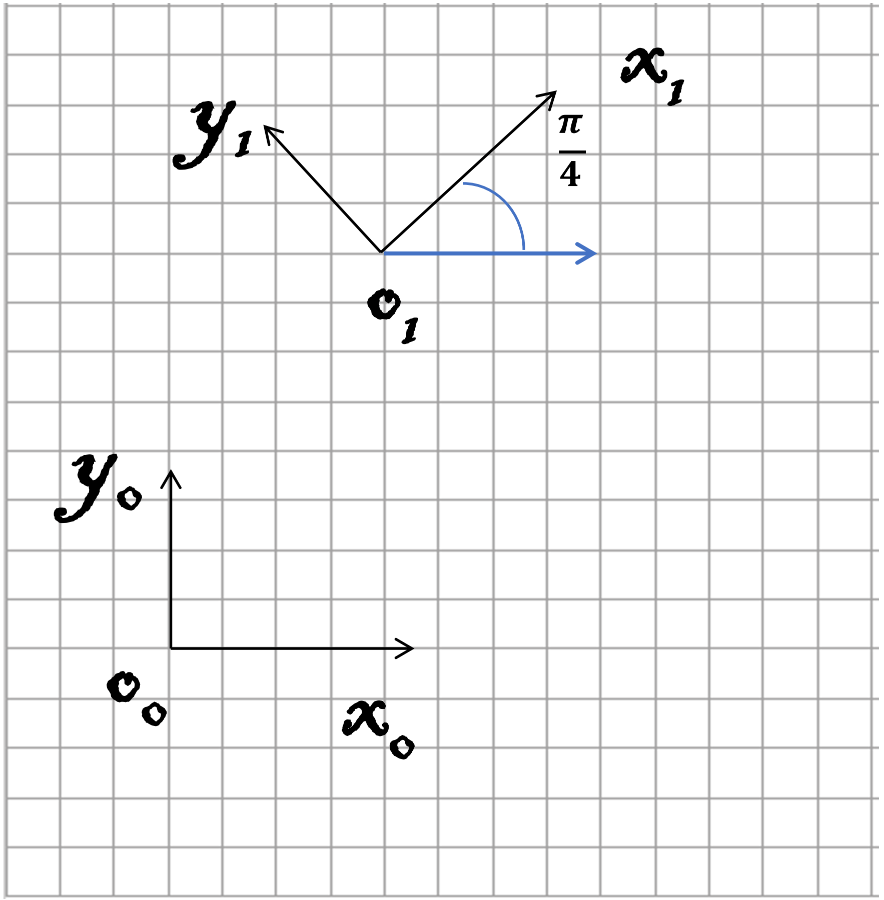
This is just our eqn from the previous page

$$\boxed{\begin{bmatrix} \boldsymbol{P}^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^0 \boldsymbol{P}^1 + \boldsymbol{d}^0 \\ 1 \end{bmatrix}} = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{d}^0 \\ 0_2 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^1 \\ 1 \end{bmatrix}$$

in which $0_2 = [0 \quad 0]$

The set of matrices of the form $\begin{bmatrix} R & d \\ 0_n & 1 \end{bmatrix}$, where $R \in SO(n)$ and $d \in \mathbb{R}^n$ is called the ***Special Euclidean Group of order*** $\boldsymbol{n}$, or $SE(n)$.

Lets practice...



$$T_1^0 = \begin{bmatrix} & & \\ & & \end{bmatrix}$$

$$T_0^1 = \begin{bmatrix} & & \\ & & \end{bmatrix}$$

Recall: $\cos\frac{\pi}{4} = 0.5\sqrt{2}$, $\quad \sin\frac{\pi}{4} = 0.5\sqrt{2}$

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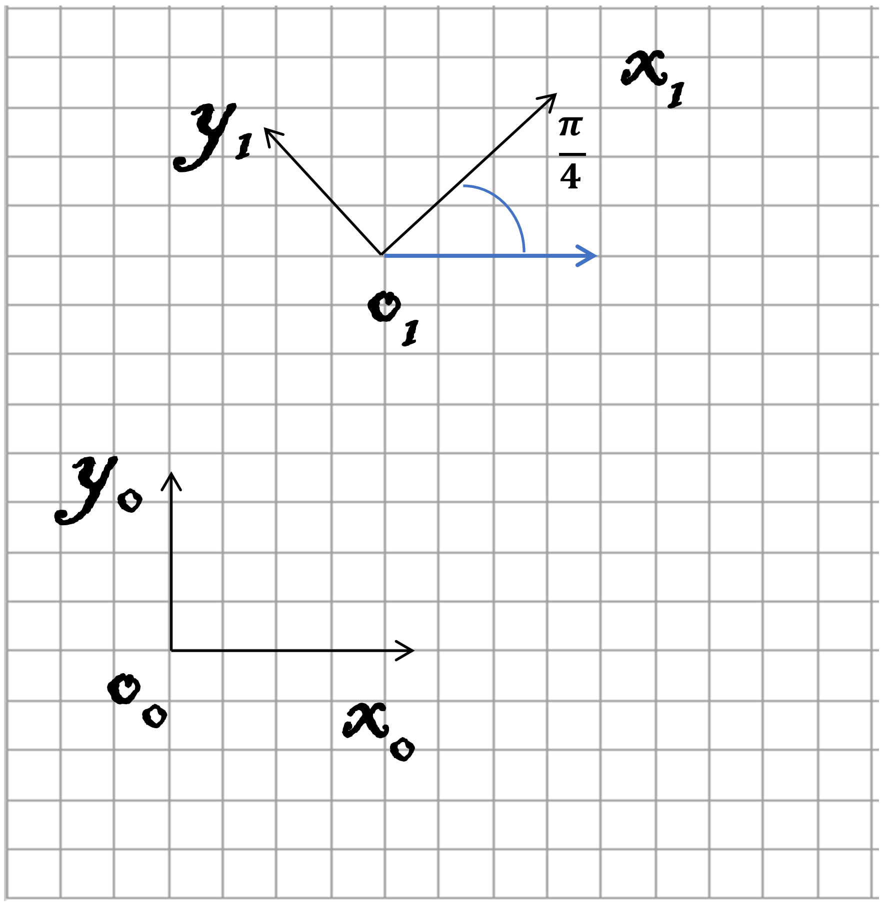


$$T_1^0 = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} & 4 \\ 0.5\sqrt{2} & 0.5\sqrt{2} & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_0^1 = \begin{bmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} & -6\sqrt{2} \\ -0.5\sqrt{2} & 0.5\sqrt{2} & -2\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Recall: $\cos\frac{\pi}{4} = 0.5\sqrt{2}$, $\sin\frac{\pi}{4} = 0.5\sqrt{2}$

Inverse of a Homogeneous Transformation

What is the relationship between T_1^0 and T_0^1?

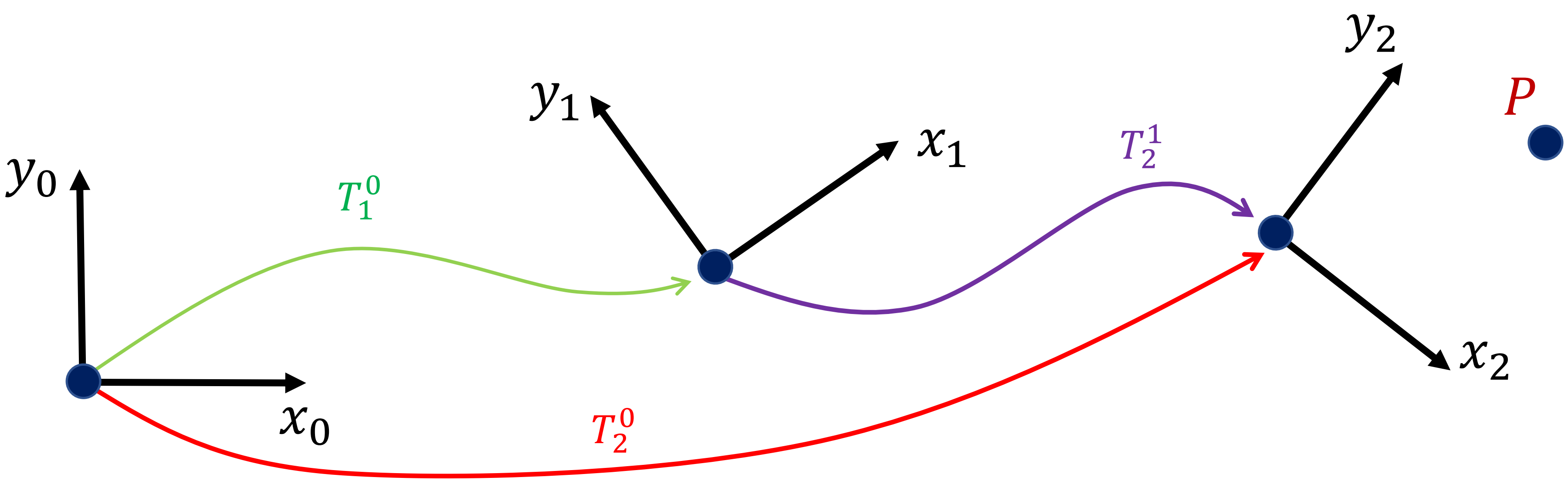
$$T_1^0 T_0^1 = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} & 4 \\ 0.5\sqrt{2} & 0.5\sqrt{2} & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} & -6\sqrt{2} \\ -0.5\sqrt{2} & 0.5\sqrt{2} & -2\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In general, $T_k^j = \left(T_j^k\right)^{-1}$ and $\begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0_n & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{d} \\ 0_n & 1 \end{bmatrix}$

This is easy to verify:

$$\begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0_n & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{d} \\ 0_n & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}\boldsymbol{R}^T & -\boldsymbol{R}\boldsymbol{R}^T\boldsymbol{d} + \boldsymbol{d} \\ 0_n & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{n\times n} & \boldsymbol{0}_n \\ 0_n & 1 \end{bmatrix} = I_{(n+1)\times(n+1)}$$

Composition of Transformations



From our previous results, we know:

$$P^0 = T_1^0 P^1$$
$$P^1 = T_2^1 P^2$$

$$P^0 = T_1^0 T_2^1 P^2$$

But we also know: $P^0 = T_2^0 P^2$

$$T_2^0 = T_1^0 T_2^1$$

This is the composition law for homogeneous transformations.