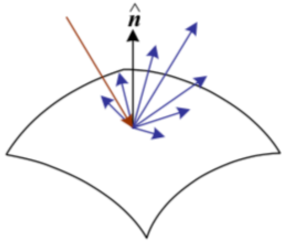
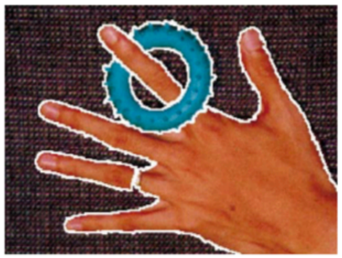


2. Image Formation

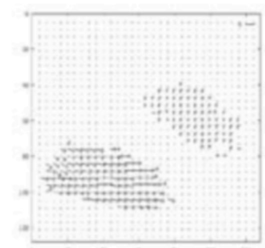
3. Image Processing

4. Features

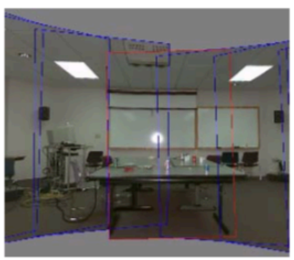


5. Segmentation

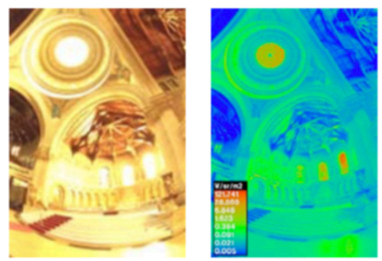
6-7. Structure from Motion



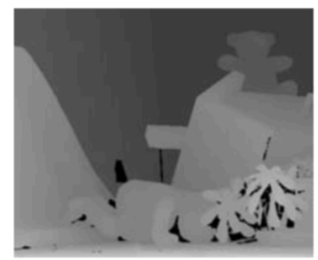
8. Motion



9. Stitching

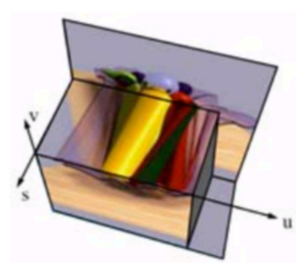


10. Computational Photography



11. Stereo

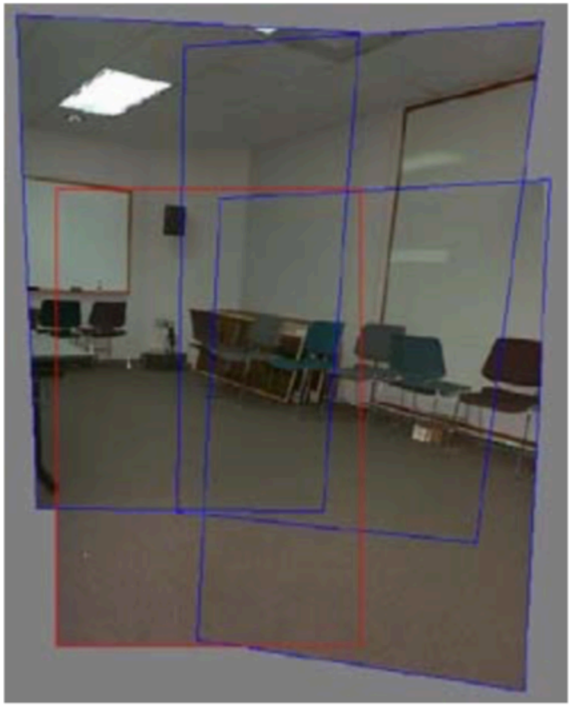
12. 3D Shape

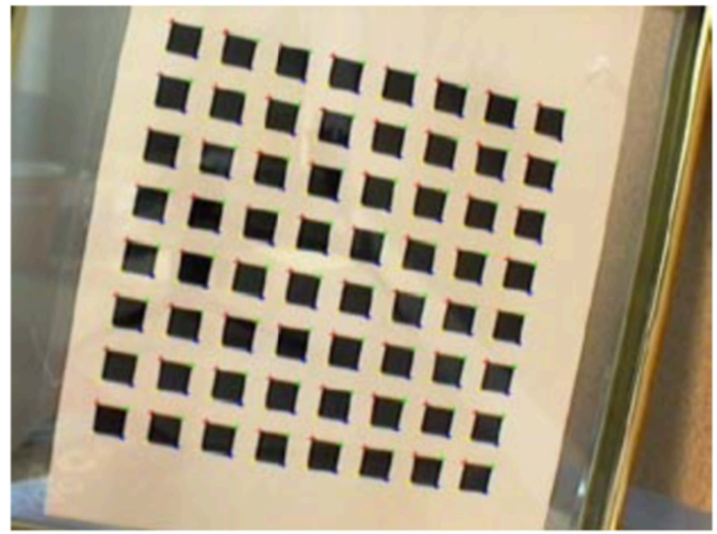


13. Image-based Rendering

14. Recognition

Feature-based Image Alignment





- Geometric image registration
  - 2D or 3D transforms between them
  - Special cases: pose estimation, calibration

Image credit Szeliski book

2D Alignment

- 3 photos
- Translational model

2D Alignment

- Input:
  - A set of matches $\{(x_i, x_i')\}$
  - A parametric model $f(x; p)$
- Output:
  - Best model p^*
- How?

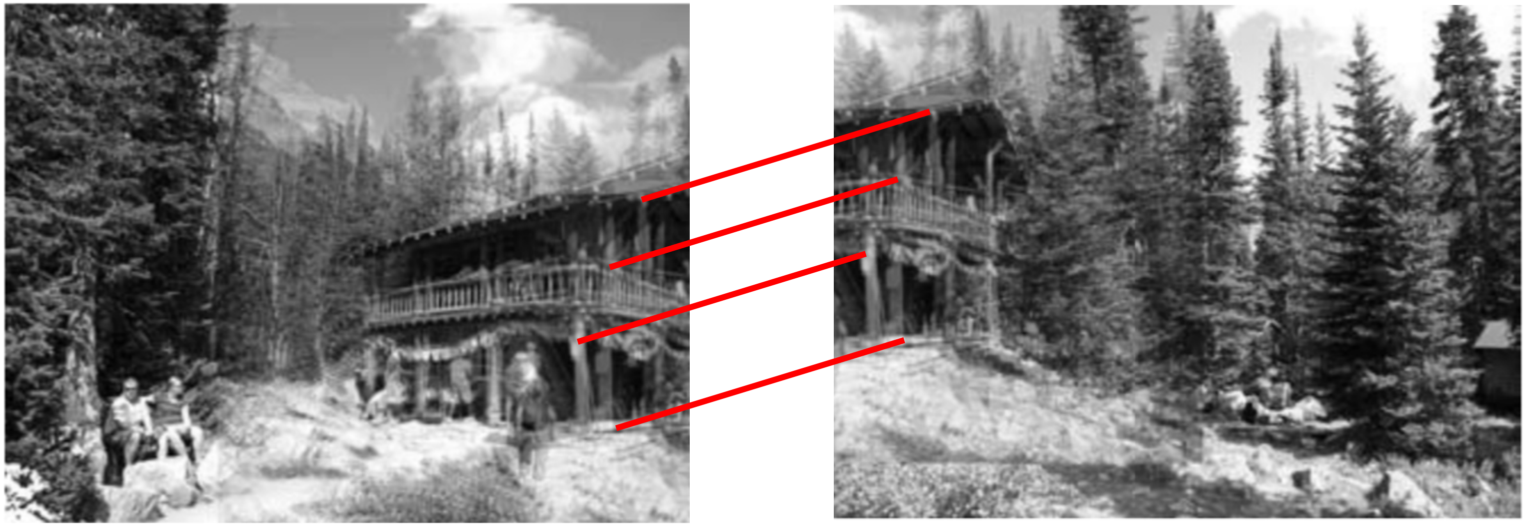
2D translation estimation

- Input:
  - Set of matches $\{(x_1, x_1'), (x_2, x_2'), (x_3, x_3'), (x_4, x_4')\}$
  - Parametric model: $f(x; t) = x + t$
  - Parameters p == t, location of origin of A in B
- Output:
  - Best model p*

2D translation estimation

- Input:
  - Set of matches $\{(x_1, x_1'), (x_2, x_2'), (x_3, x_3'), (x_4, x_4')\}$
  - Parametric model: $f(x; t) = x + t$
  - Parameters p == t, location of origin of A in B
- Question for class:
  - What is your best guess for model p* ??

2D translation estimation



- How?
  - One correspondence x1 = [600, 150], x1' = [50, 50]
  - Parametric model: x' = f(x; t) = x + t

2D translation estimation

- How?
  - One correspondence $x1 = [600, 150]$, $x1' = [50, 50]$
  - Parametric model: $x' = f(x; t) = x + t$

    => $t = x' - x$

    => $t = [50\text{-}600, 40\text{-}150] = [-550, -100]$

2D translation via least-squares

- How?
  - A set of matches $\{(x_i, x_i')\}$
  - Parametric model: $f(x; t) = x + t$
  - Minimize sum of squared residuals:

$$E_{\mathrm{LS}} = \sum_i \|\boldsymbol{r}_i\|^2 = \sum_i \|\boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}_i'\|^2,$$

How to solve?

In many cases, parametric model is linear:

Jacobian

$$f(x;p) = x + J(x)p$$

$$\Delta x = x' - x = J(x)p$$

$$E_{LS} = \sum_i \|J(x)p + x - x_i'\|^2 = \sum_i \|J(x_i)p - \Delta x_i\|^2$$

Differentiate and set to 0:

$$2\sum_i J^T(x_i)\left(J(x_i)p - \Delta x_i\right) = 0$$

Normal equations —

$$\left[\sum_i J^T(x_i)J(x_i)\right] p = \sum_i J^T(x_i)\Delta x_i$$

$$Ap = b$$

Hessian

$$p* = A^{-1}b$$

Linear models menagerie

Transform	Matrix	Parameters p	Jacobian J
translation	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix}$	(t_x, t_y)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Euclidean	$\begin{bmatrix} c_\theta & -s_\theta & t_x \\ s_\theta & c_\theta & t_y \end{bmatrix}$	(t_x, t_y, θ)	$\begin{bmatrix} 1 & 0 & -s_\theta x - c_\theta y \\ 0 & 1 & c_\theta x - s_\theta y \end{bmatrix}$
similarity	$\begin{bmatrix} 1+a & -b & t_x \\ b & 1+a & t_y \end{bmatrix}$	(t_x, t_y, a, b)	$\begin{bmatrix} 1 & 0 & x & -y \\ 0 & 1 & y & x \end{bmatrix}$
affine	$\begin{bmatrix} 1+a_{00} & a_{01} & t_x \\ a_{10} & 1+a_{11} & t_y \end{bmatrix}$	$(t_x, t_y, a_{00}, a_{01}, a_{10}, a_{11})$	$\begin{bmatrix} 1 & 0 & x & y & 0 & 0 \\ 0 & 1 & 0 & 0 & x & y \end{bmatrix}$

- All the simple 2D models are linear!
- Exception: perspective transform

2D translation via least-squares

For translation: $J = I$ and normal equations are particularly simple:

$$\left[\sum_i I^T I\right] p = \sum_i \Delta x_i$$

$$p* = \frac{1}{n} \sum_i \Delta x_i$$

In other words: just **average** the "flow vectors" $\Delta x = x' - x$

Image credit Szeliski book

Oops I lied !!! Euclidean is not linear!

Transform	Matrix	Parameters p	Jacobian J
translation	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix}$	(t_x, t_y)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Euclidean	$\begin{bmatrix} c_\theta & -s_\theta & t_x \\ s_\theta & c_\theta & t_y \end{bmatrix}$	(t_x, t_y, θ)	$\begin{bmatrix} 1 & 0 & -s_\theta x - c_\theta y \\ 0 & 1 & c_\theta x - s_\theta y \end{bmatrix}$
similarity	$\begin{bmatrix} 1+a & -b & t_x \\ b & 1+a & t_y \end{bmatrix}$	(t_x, t_y, a, b)	$\begin{bmatrix} 1 & 0 & x & -y \\ 0 & 1 & y & x \end{bmatrix}$
affine	$\begin{bmatrix} 1+a_{00} & a_{01} & t_x \\ a_{10} & 1+a_{11} & t_y \end{bmatrix}$	$(t_x, t_y, a_{00}, a_{01}, a_{10}, a_{11})$	$\begin{bmatrix} 1 & 0 & x & y & 0 & 0 \\ 0 & 1 & 0 & 0 & x & y \end{bmatrix}$

- ~~All the simple 2D models are linear!~~
- Euclidean Jacobians are a function of θ!

Figure credit Szeliski book

Nonlinear Least Squares

$$E_{NLS} = \sum_i \|f(x_i; p) - x'_i\|^2$$

Linearize around a current guess p:

$$f(x; p + \Delta p) = f(x; p) + J(x; p)\Delta p$$

$$r = x' - f(x; p) = J(x; p)\Delta p$$

$$E_{NLS} = \sum_i \|f(x; p) + J(x; p)\Delta p - x'_i\|^2 = \sum_i \|J(x; p)\Delta p - r_i\|^2$$

Differentiate and set to 0:

$$2\sum_i J^T(x_i; p)\left(J(x_i; p)\Delta p - r_i\right) = 0$$

$$\left[\sum_i J^T(x_i; p) J(x_i; p)\right]\Delta p = \sum_i J^T(x_i; p) r_i$$

$$A\Delta p = b$$

$$\Delta p* = A^{-1} b$$

Projective/H

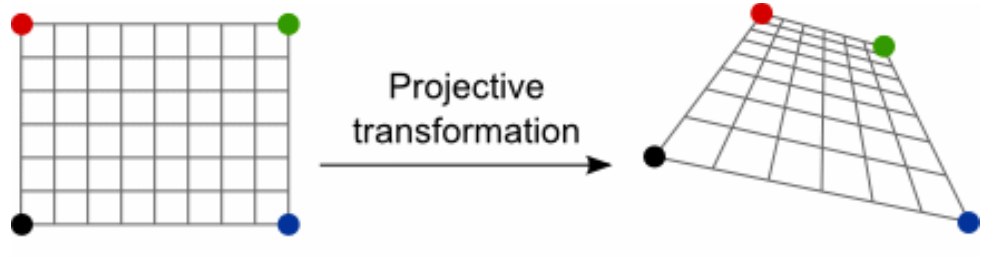


Image credit Graphics Mill (educational Use)

- Jacobians a bit harder
- Parameterization:

$$\begin{bmatrix} 1+h_{00} & h_{01} & h_{02} \\ h_{10} & 1+h_{11} & h_{12} \\ h_{20} & h_{21} & 1 \end{bmatrix} \qquad (h_{00}, h_{01}, \ldots, h_{21})$$

- x'= f(x,p):

$$x' = \frac{(1+h_{00})x + h_{01}y + h_{02}}{h_{20}x + h_{21}y + 1} \text{ and } y' = \frac{h_{10}x + (1+h_{11})y + h_{12}}{h_{20}x + h_{21}y + 1}.$$

- And Jacobian:

$$\boldsymbol{J} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}} = \frac{1}{D} \begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -x'x & -x'y \\ 0 & 0 & 0 & x & y & 1 & -y'x & -y'y \end{bmatrix}$$

$$D = h_{20}x + h_{21}y + 1$$

Closed Form H

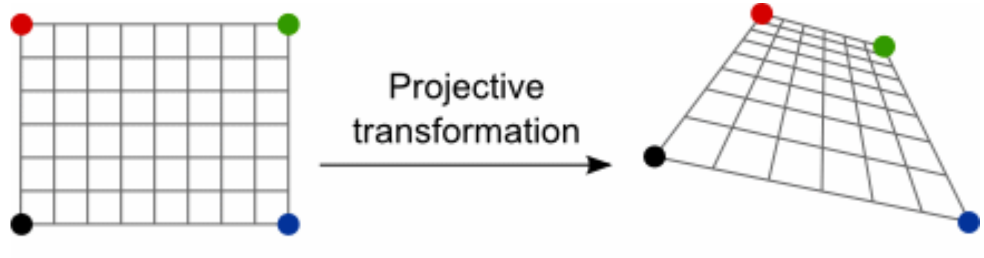


Image credit Graphics Mill

- Taking x'=f(x,p):

$$x' = \frac{(1+h_{00})x + h_{01}y + h_{02}}{h_{20}x + h_{21}y + 1} \text{ and } y' = \frac{h_{10}x + (1+h_{11})y + h_{12}}{h_{20}x + h_{21}y + 1}.$$

- Divide both sides by $D = h_{20}x + h_{21}y + 1$:

$$\begin{bmatrix} \hat{x}' - x \\ \hat{y}' - y \end{bmatrix} = \begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -\hat{x}'x & -\hat{x}'y \\ 0 & 0 & 0 & x & y & 1 & -\hat{y}'x & -\hat{y}'y \end{bmatrix} \begin{bmatrix} h_{00} \\ \vdots \\ h_{21} \end{bmatrix}$$

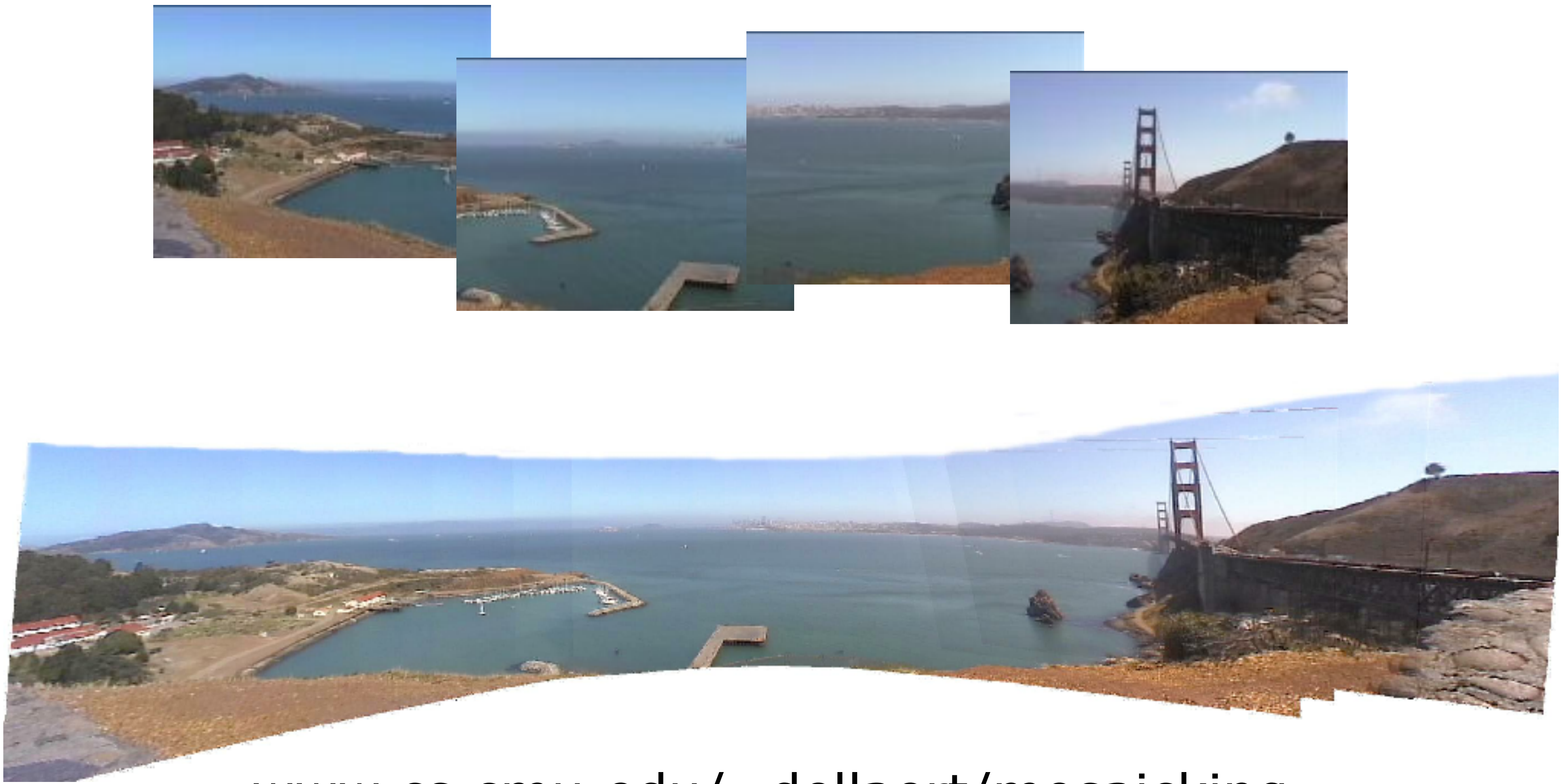
- 4 matches => system of 8 linear equations

RANSAC

Motivation

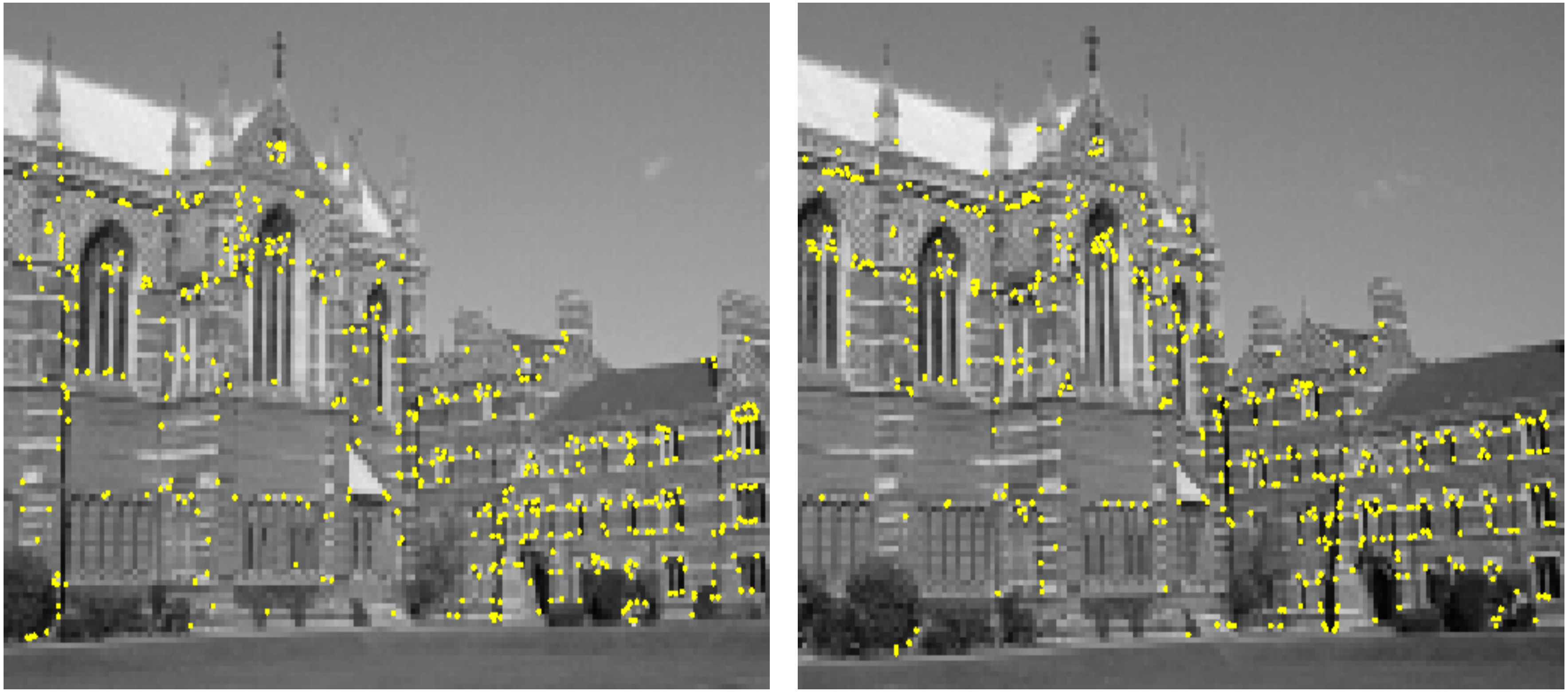
- Estimating motion models
- Typically: points in two images
- Candidates:
  - Translation
  - Homography
  - Fundamental matrix

Mosaicking: Homography

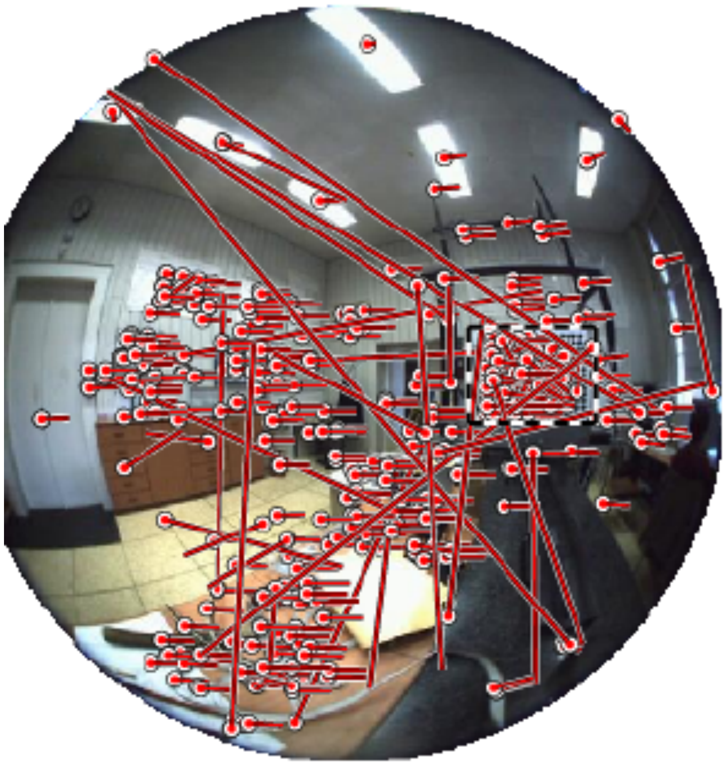


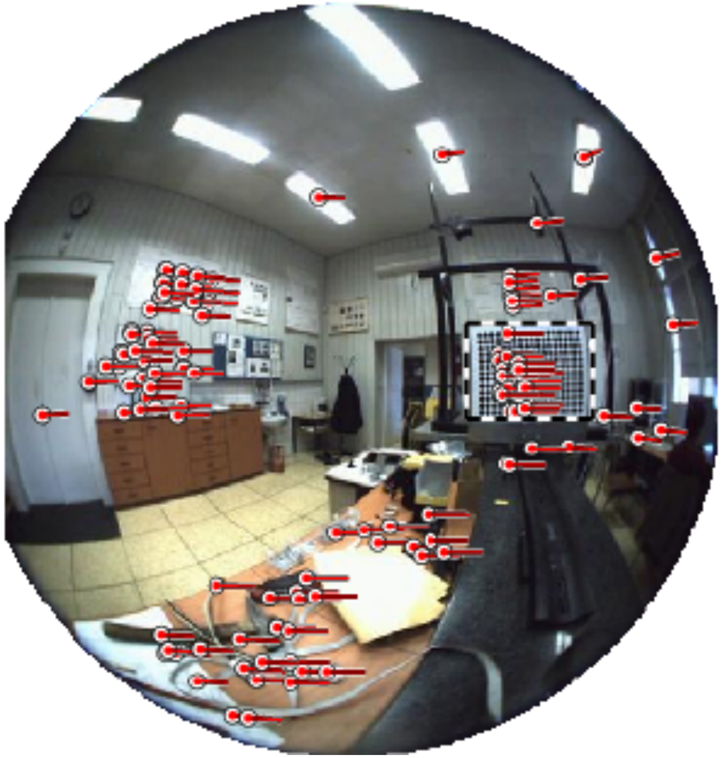
www.cs.cmu.edu/~dellaert/mosaicking

Two-view geometry (next lecture)



Omnidirectional example

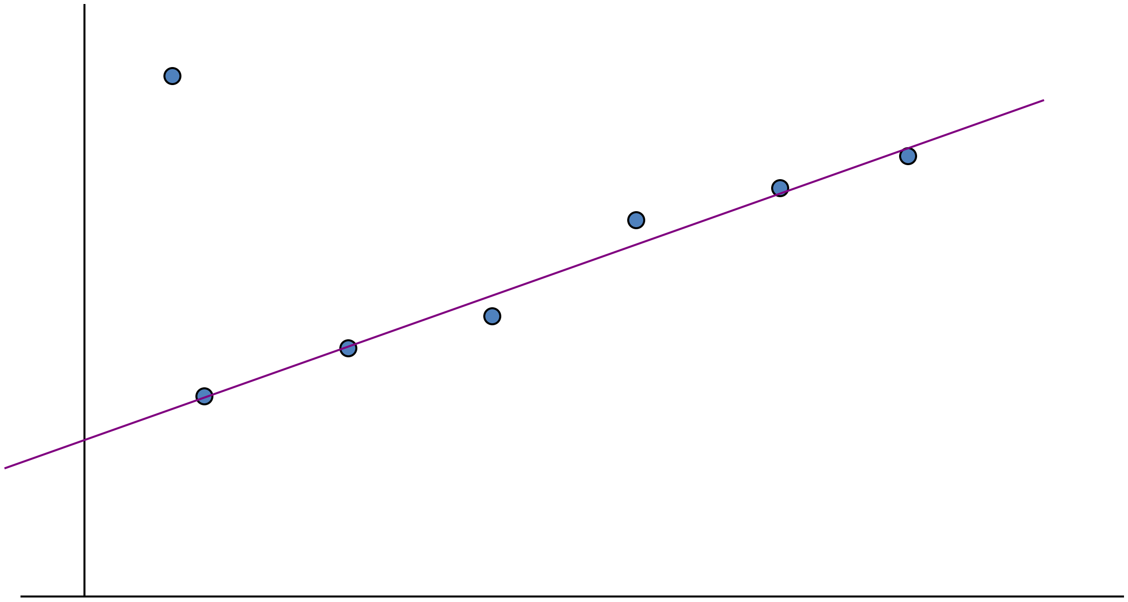




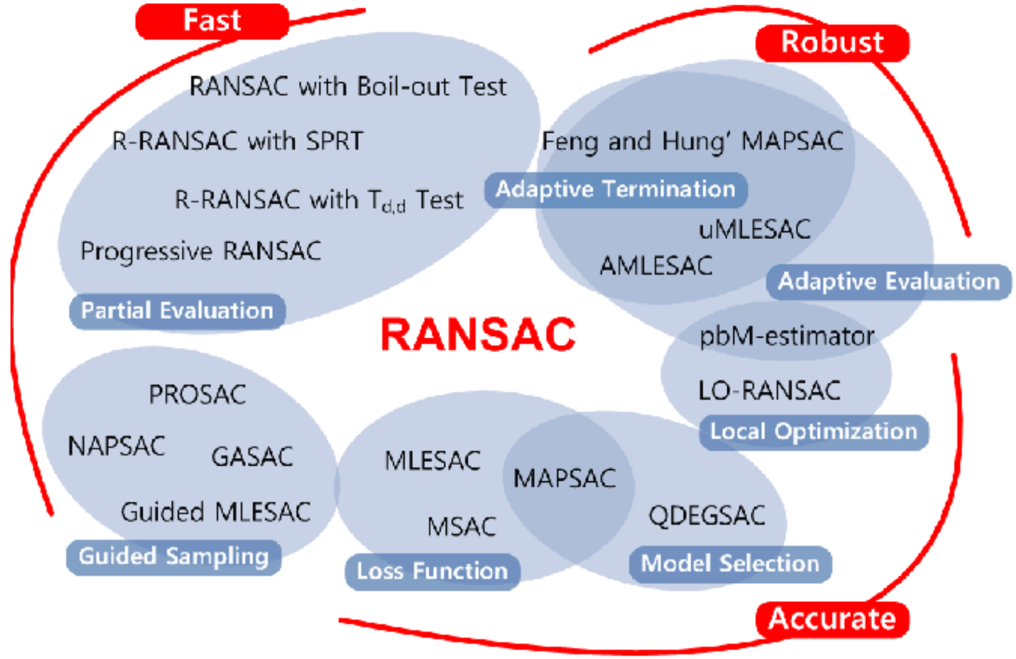
Images by Branislav Micusik, Tomas Pajdla,
cmp.felk.cvut.cz/ demos/Fishepip/

Simpler Example

- Fitting a straight line



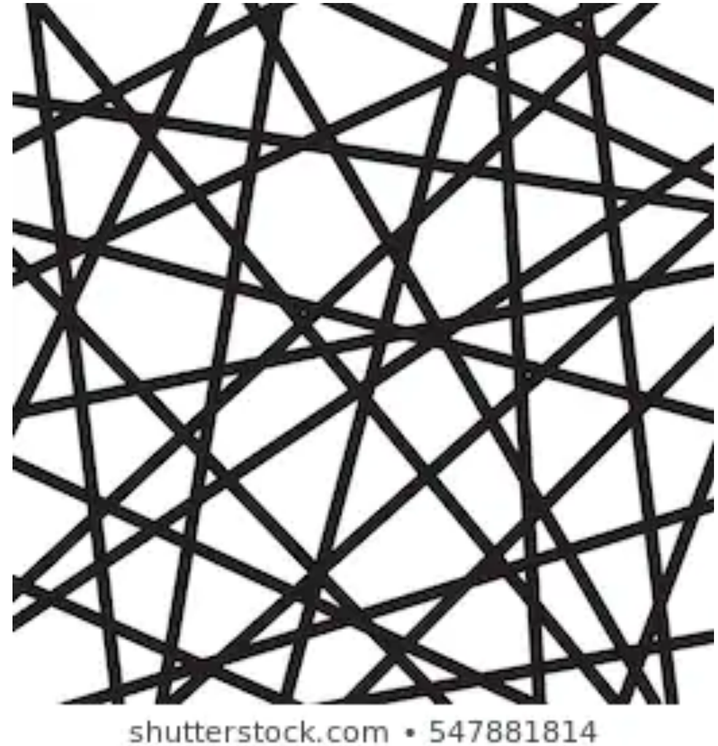
Discard Outliers



- No point with d>t
- RANSAC:
  – RANdom SAmple Consensus
  – Fischler & Bolles 1981
  – Copes with a large proportion of outliers

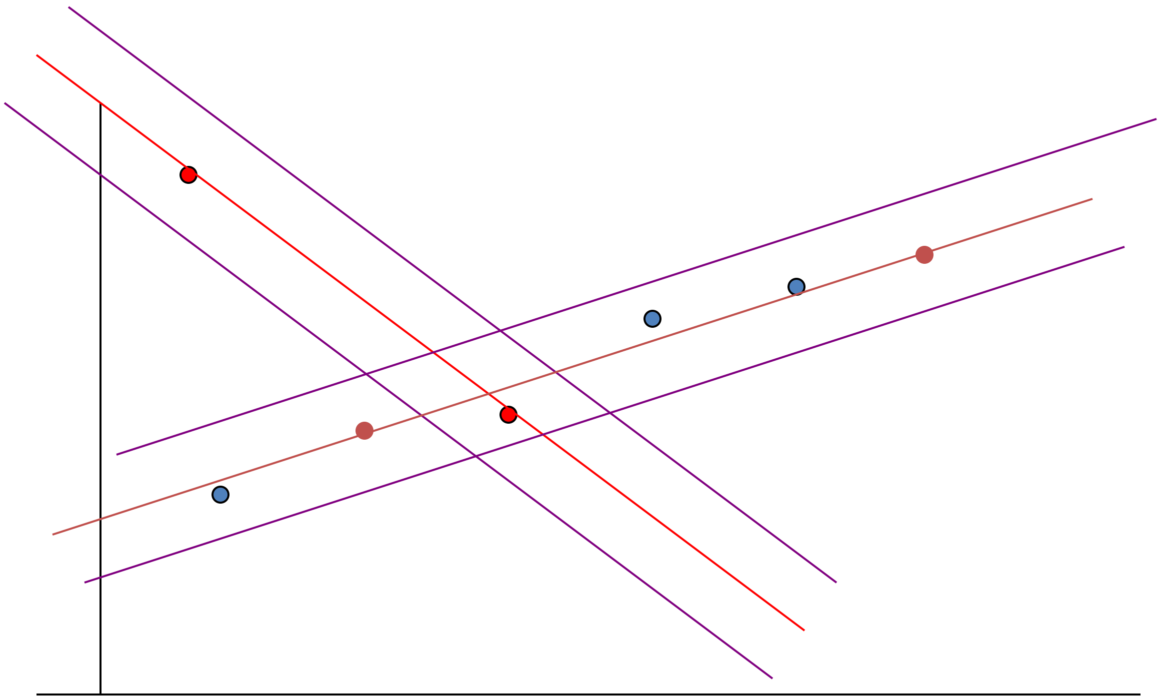
Image credit Choi et al BMVC 2009

Main Idea



- Select 2 points at random
- Fit a line
- “Support” = number of inliers
- Line with most inliers wins

Why will this work ?



Best Line has most support

- More support -> better fit

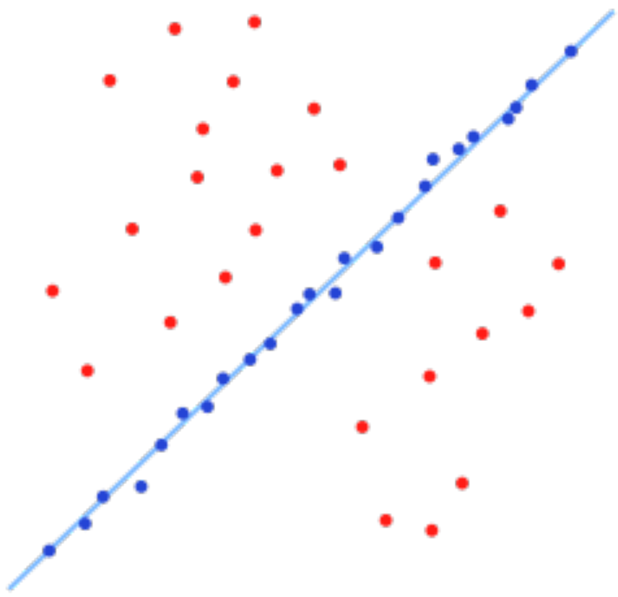
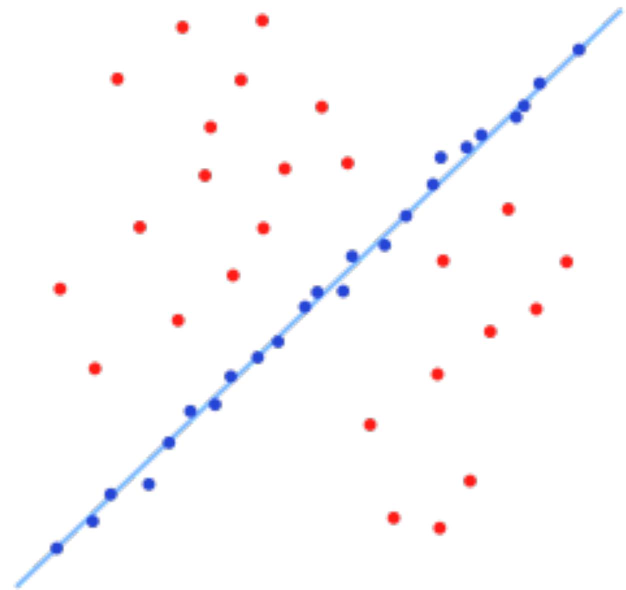


Image credit Wikipedia

RANSAC



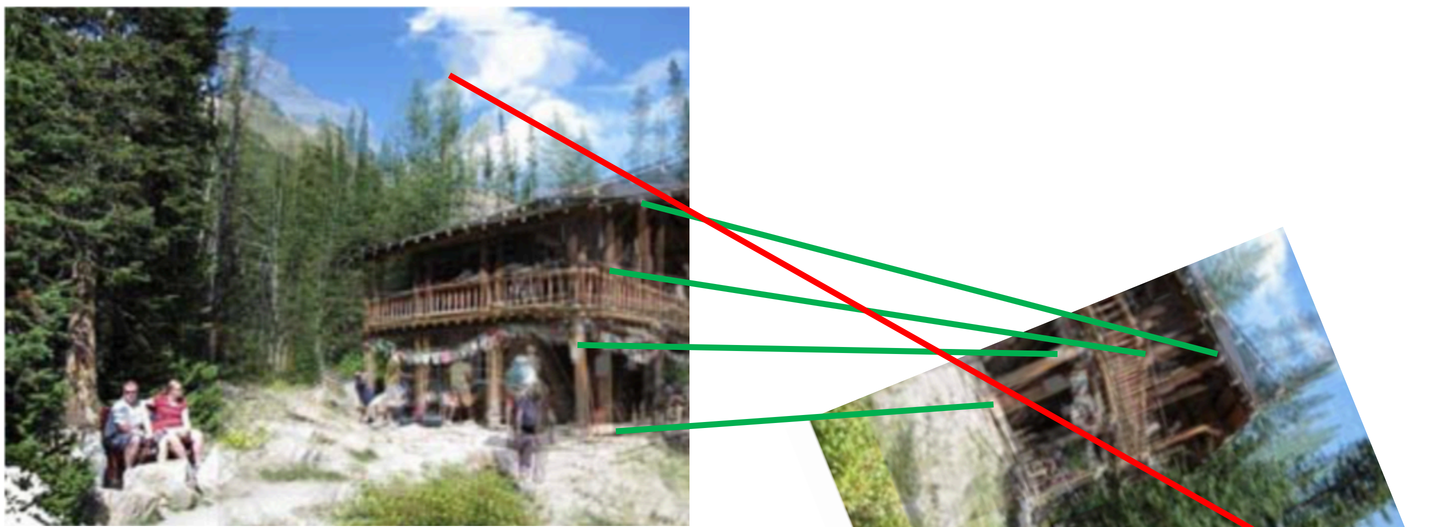
- Objective:
  - Robust fit of a model to data D
- Algorithm
  - Randomly select s points
  - Instantiate a model
  - Get consensus set D_i
  - If $|D_i|>T$, terminate and return model
  - Repeat for N trials, return model with max $|D_i|$

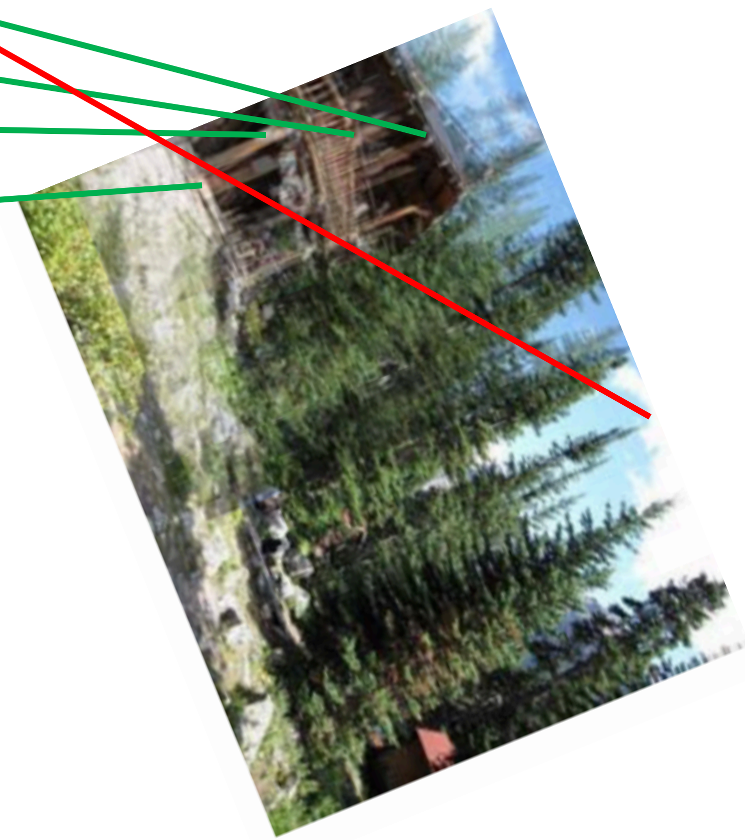
Image credit Wikipedia

In General

- Fit a more general model
- Sample = minimal subset
  - Translation ?
  - Homography ?
  - Euclidean transorm ?

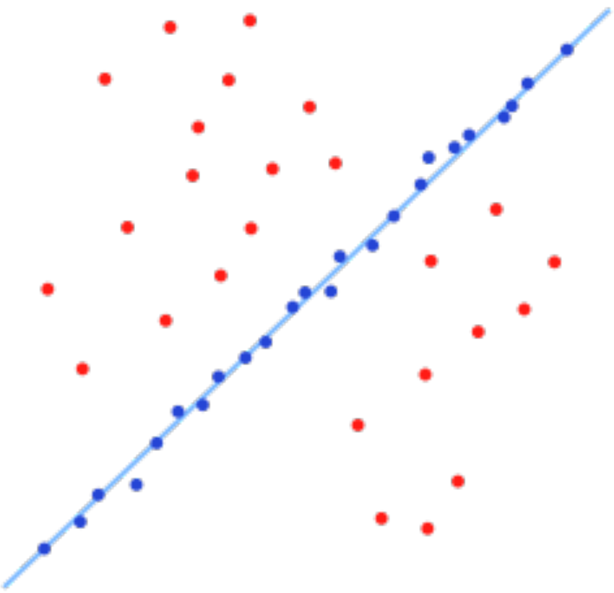
Example





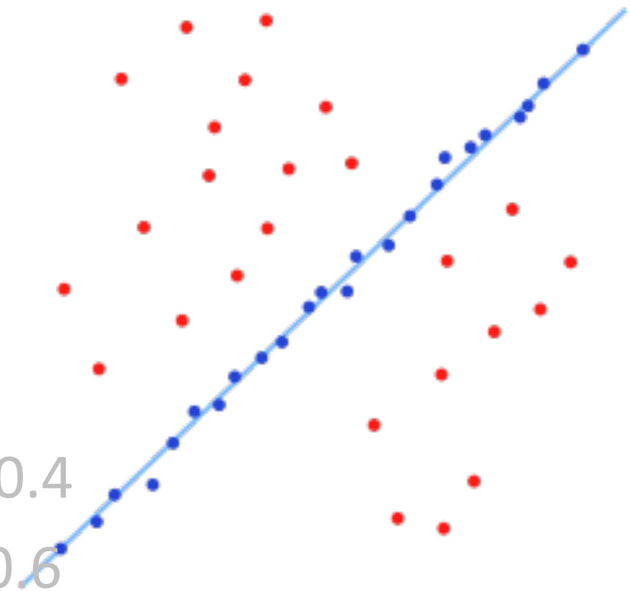
- Euclidean: needs 2 correspondences (2*2>=3)
- Here correct hypothesis has support of 4 (out of 5)
- Including red into minimal sample (of 2) would **likely** yield low support

How many samples ?



- We want: at least one sample with **all inliers**
- Can’t guarantee: probability P
- E.g. P = 0.99

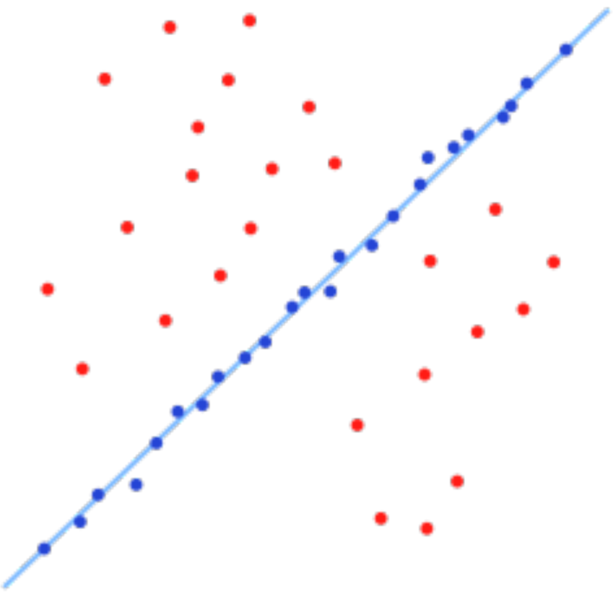
Calculate N



- If ε = outlier probability — ε =0.4
- proportion of inliers $p = 1- \varepsilon$ — p=0.6
- P(sample with all inliers) = p^s — s=2 -> p^s=0.36
- P(sample with an outlier) = $1-p^s$ — 0.64
- P(N samples an outlier) = $(1-p^s)^N$ — N=3 -> 0.26
- We want P(N samples an outlier) < 1-P (e.g. 0.01)
- $(1-p^s)^N < 1-P$ — $0.64^N < 0.01$
- $N > \log(1-P)/\log(1-p^s)$ — N >10.3

Image credit Wikipedia

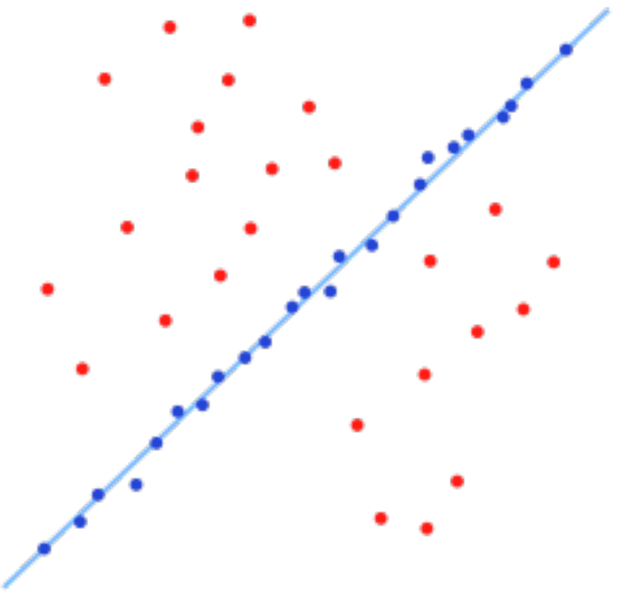
Example



- P=0.99
- s=2
  - ε = 5% => N=2
  - ε = 50% => N=17
- s=4
  - ε = 5% => N=3
  - ε = 50% => N=72
- s=8
  - ε = 5% => N=5
  - ε = 50% => N=1177

Image credit Wikipedia

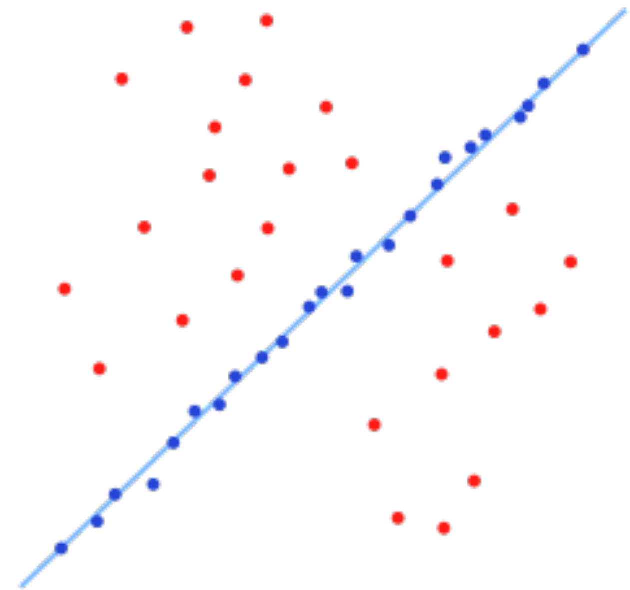
Remarks



- N = f(ε), **not** the number of points
- N increases steeply with s

Image credit Wikipedia

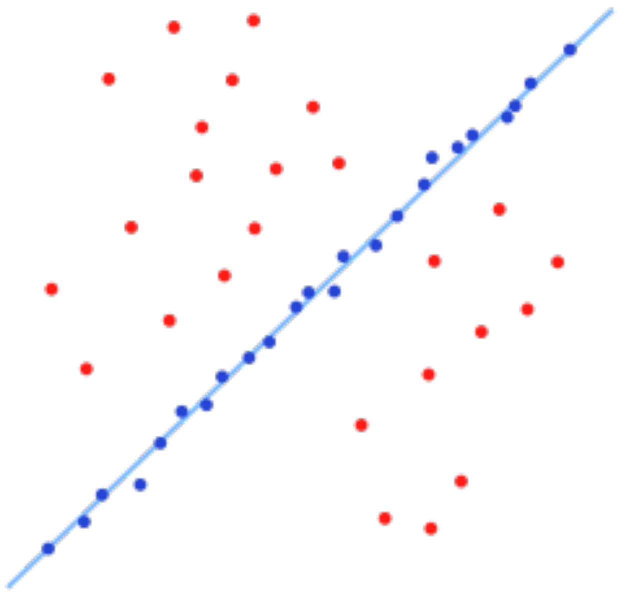
Distance Threshold



- Requires noise distribution
- Gaussian noise with σ
- Chi-squared distribution with DOF m
  - 95% cumulative:
  - Line, F: m=1, $t^2=3.84\ \sigma^2$
  - Translation, homography: m=2, $t^2=5.99\ \sigma^2$
- I.e. -> 95% prob that d<t is inlier

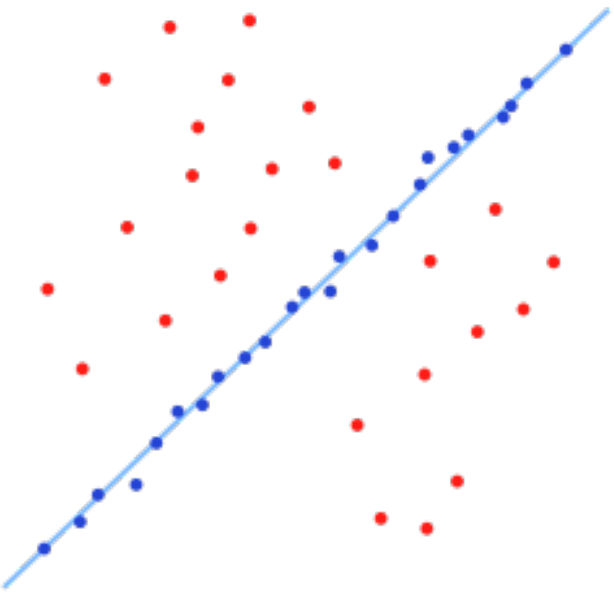
Image credit Wikipedia

Threshold T



- Terminate if $|D_i|>T$
- Rule of thumb: $T \approx$ #inliers
- So, $T = (1-\varepsilon)n = pn$

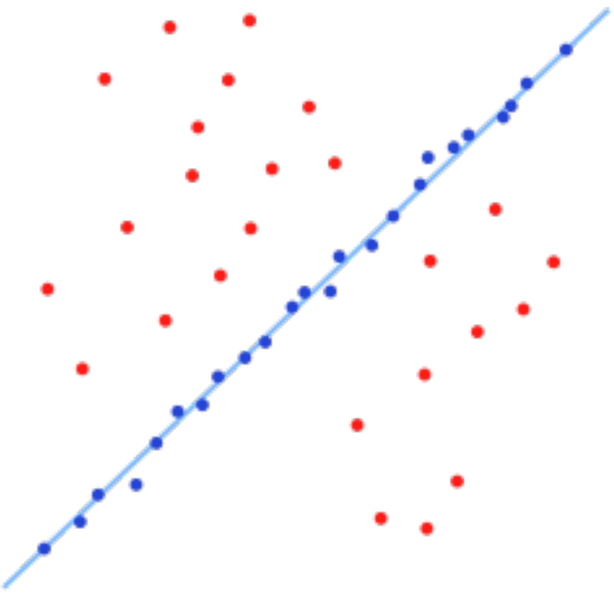
Adaptive N



- When ε is unknown ?
- Start with ε = 50%, N=inf
- Repeat:
  - Sample s, fit model
  - update ε as |outliers|/n
  - set N=f(ε, s, p)
- Terminate when N samples seen

Image credit Wikipedia

Summary: RANSAC



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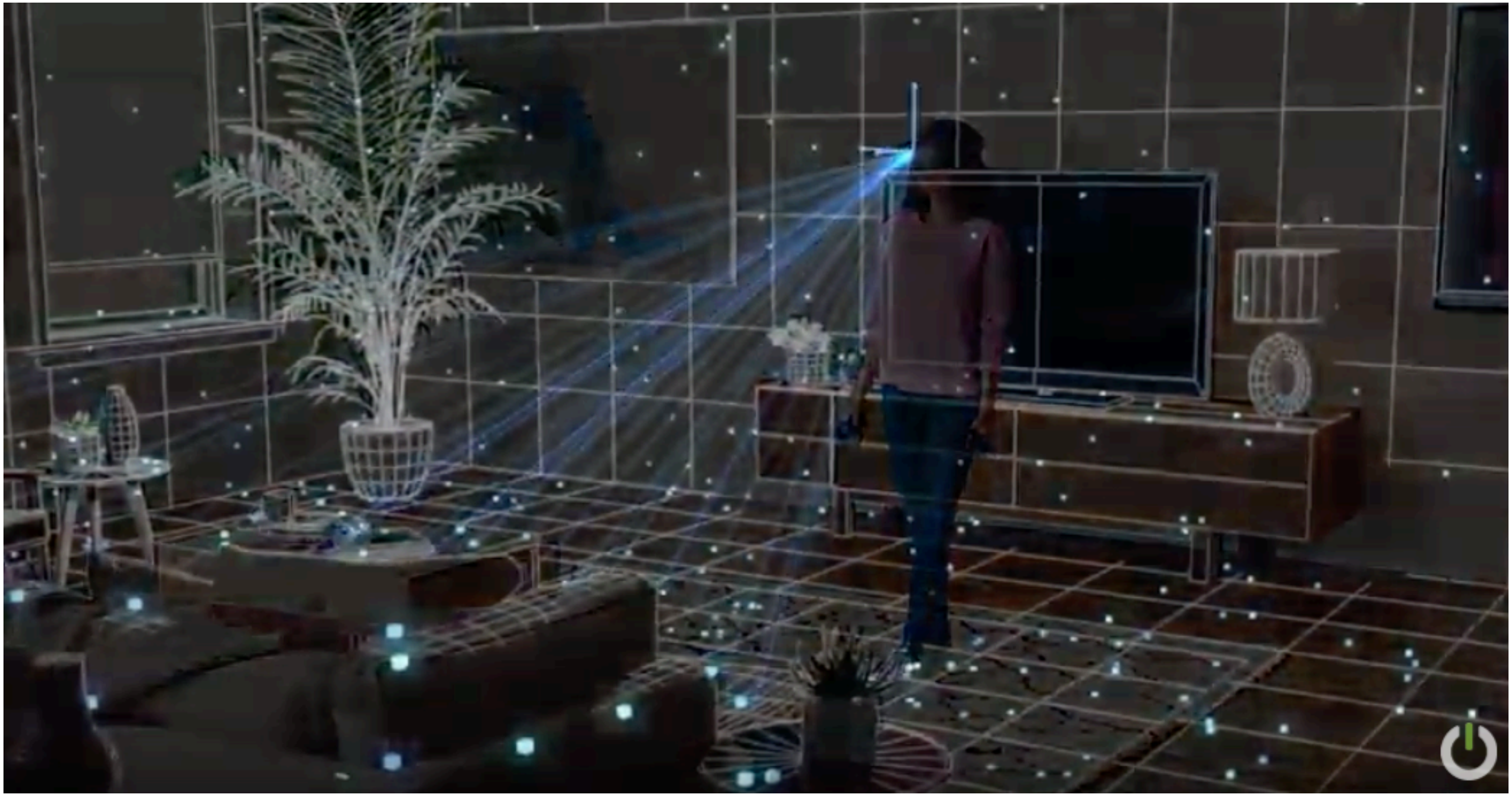
  - Randomly select s points
  - Instantiate a model
  - Get consensus set D_i
  - If $|D_i|>T$, terminate and return model
  - Repeat for N trials, return model with max $|D_i|$

Pose Estimation in VR



https://youtu.be/nrj3JE-NHMw

Review: 2D Alignment

- Input:
  - A set of matches $\{(x_i, x_i')\}$
  - A parametric model $f(x; p)$
- Output:
  - Best model p^*
- How?

Now: 3D-2D Alignment

- Input:
  - A set of 3D->2D matches $\{(X_i, x_i)\}$
  - A parametric model $f(X; p)$
- Output:
  - Best model p^*
- How?

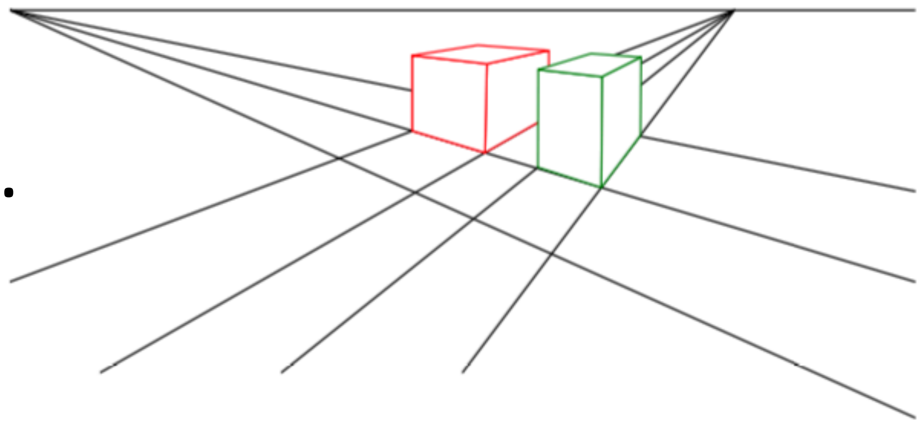
Pose Estimation

- Input:
  - A set of 2D measurements x_i of known 3D points X_i
  - Parametric model is camera matrix P, i.e., $x = f(X; P)$
- Output:
  - Best camera matrix P
- How?

Image credit Szeliski book

Review: Projective Camera Matrix

- Chapter 2 in book
- Homogeneous coord.
- 3D TO 2D projection:



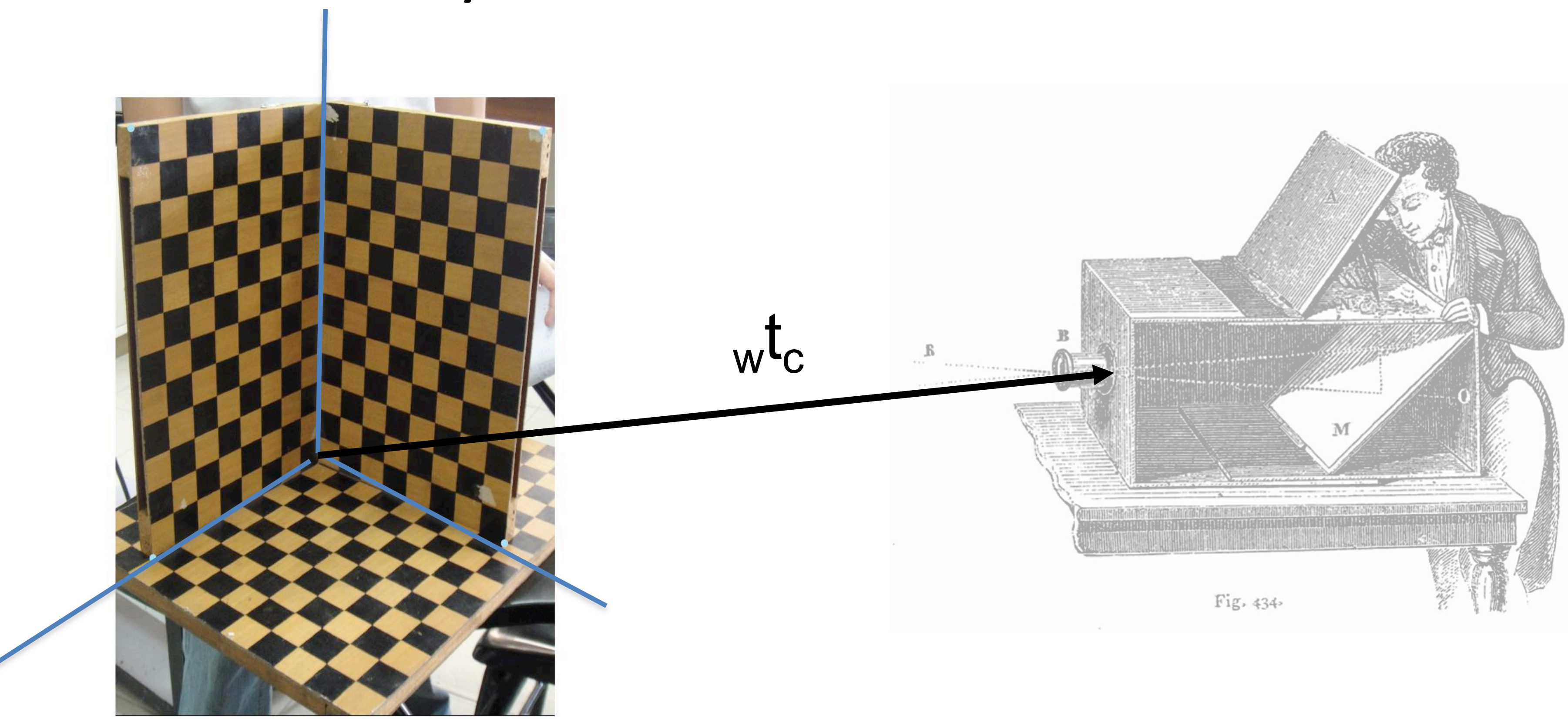
$$x = K[R|t]X = PX$$

where P = 3x4 camera matrix
and K the 3x3 calibration

$$\boldsymbol{K} = \begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

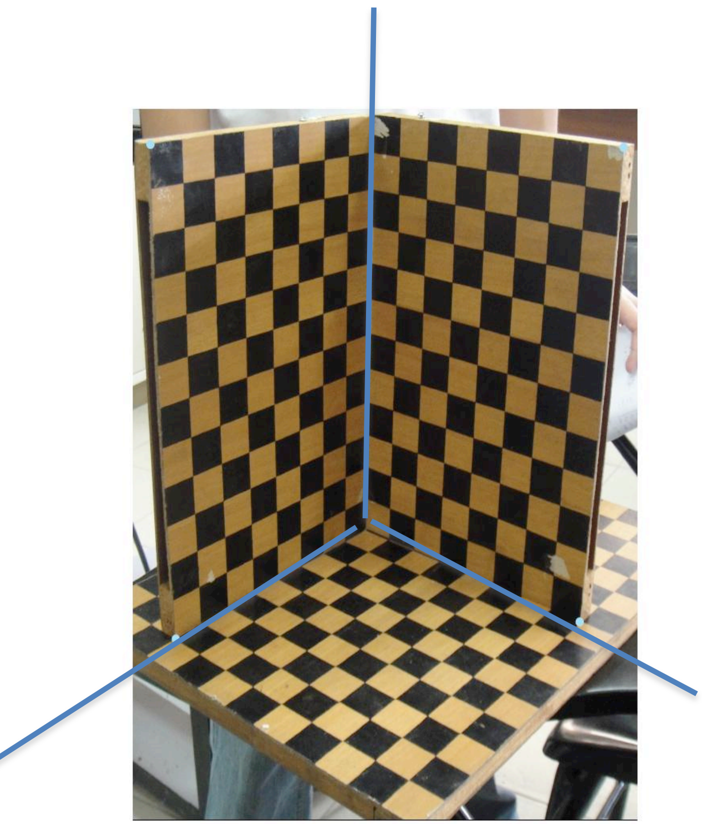
Camera Extrinsics: a Pose in 3D

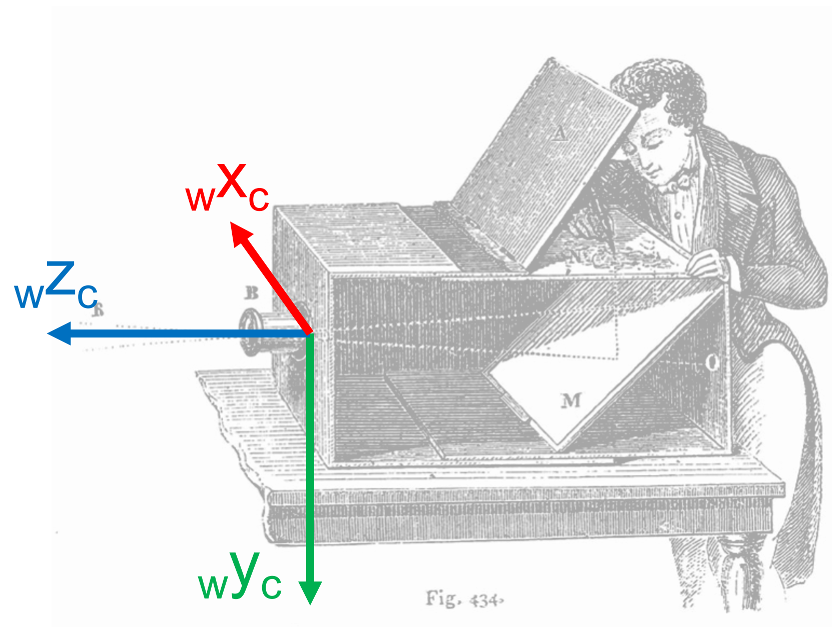
- What is the geometric meaning of R and t ??
- Intuitive: camera is at a position ${}_w t_c$
  Indices say: camera *in* world coordinate frame



Camera Extrinsics: a Pose in 3D

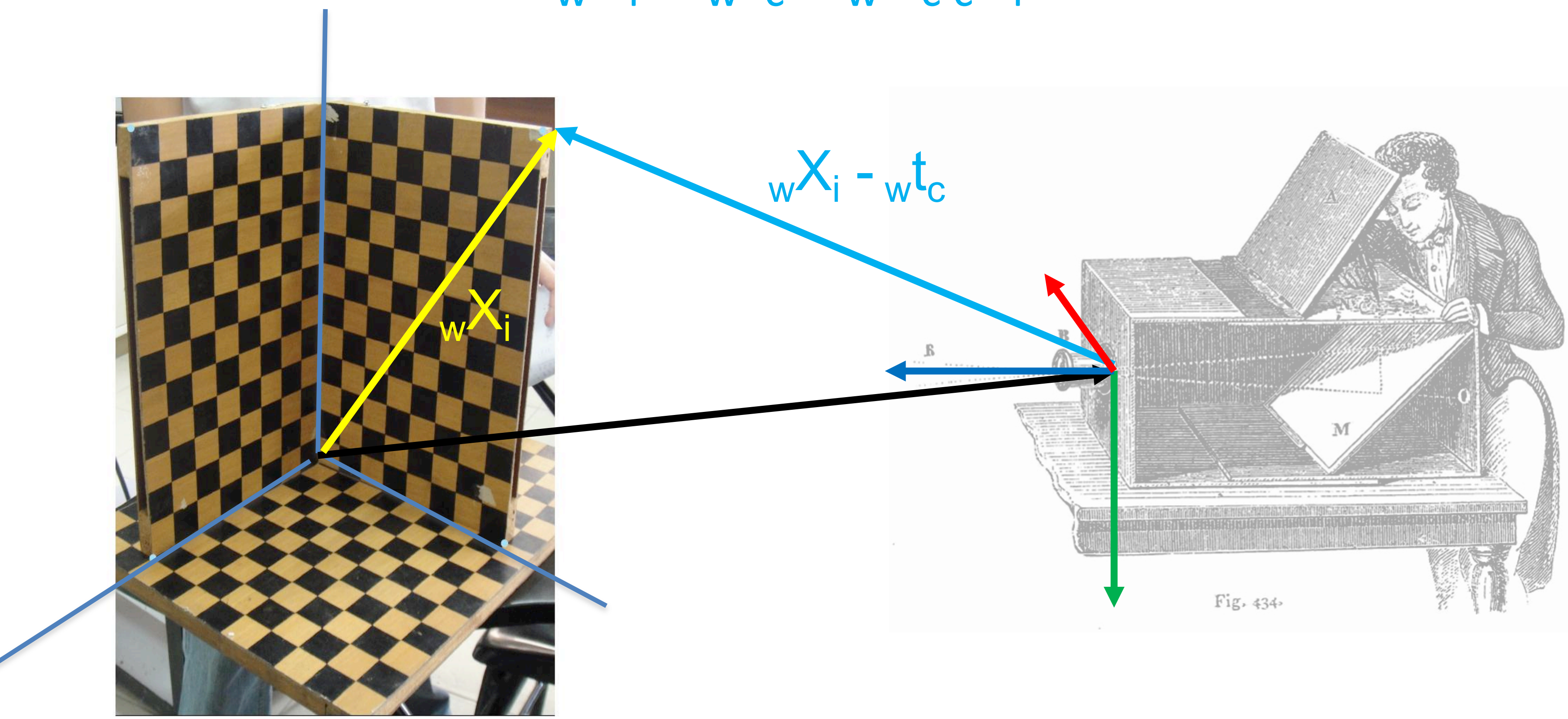
- What is the geometric meaning of R and t ??
- Rotation is given by 3x3 matrix ${}_{w}R_{c}$ whose *columns* are the camera axes ${}_{w}x_{c}$, ${}_{w}y_{c}$, ${}_{w}z_{c}$





Camera Extrinsics: a Pose in 3D

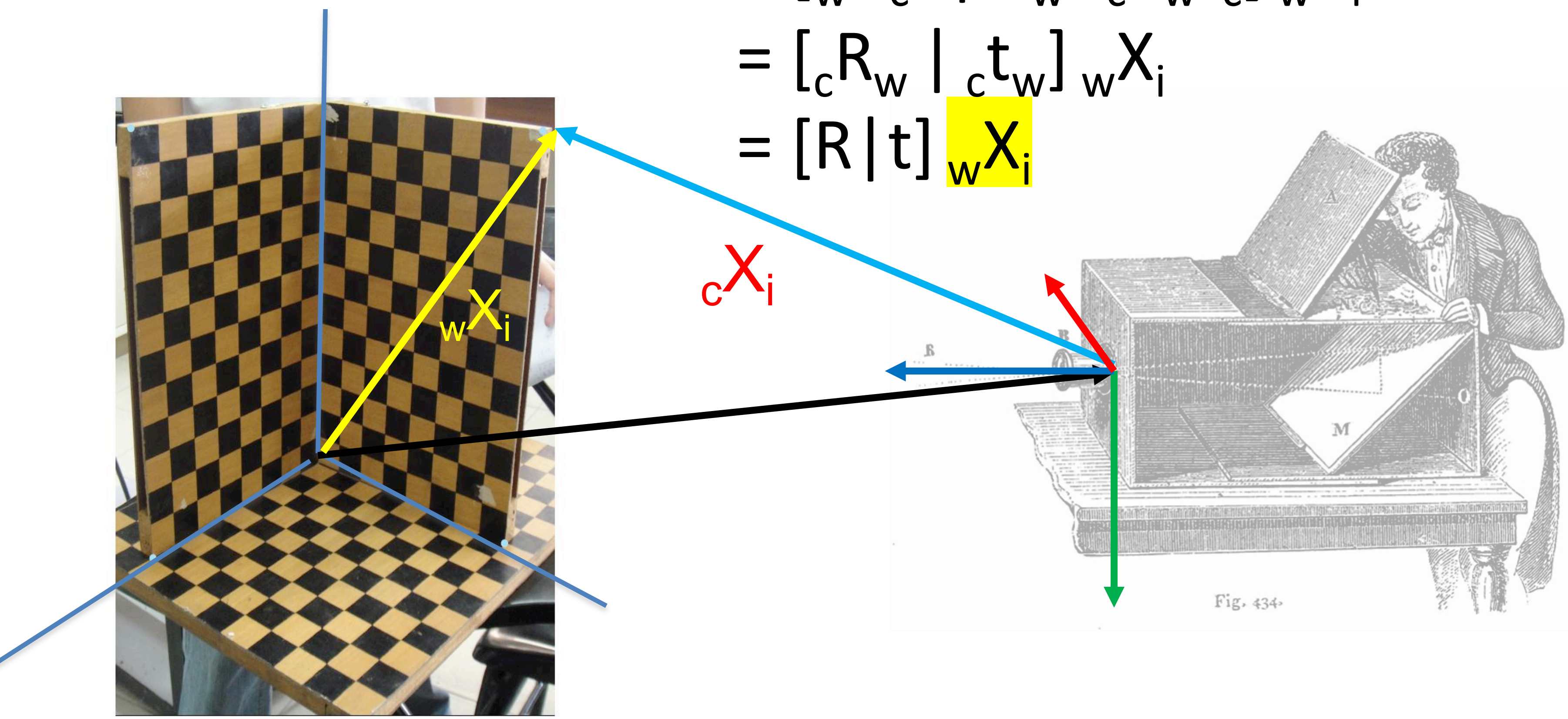
- What is the geometric meaning of R and t ??
- Transforming point X_i from world to camera coordinates: ${}_wX_i - {}_wt_c = {}_wR_c\, {}_cX_i$



Camera Extrinsics: a Pose in 3D

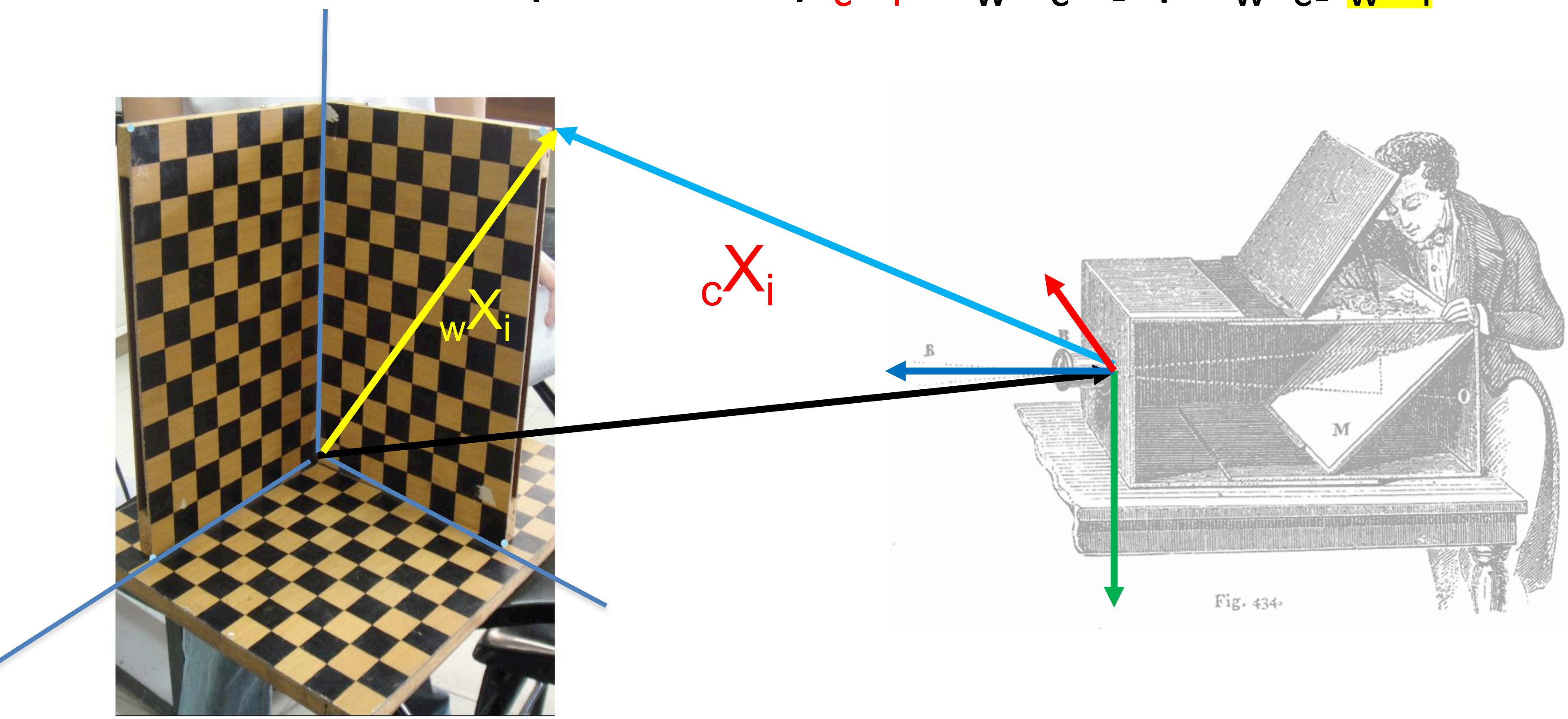
- Expressed in homogeneous coordinates:
- $_cX_i = {_wR_c}^T ({_wX_i} - {_wt_c}) = {_wR_c}^T [I| - {_wt_c}]\, {_wX_i}$

$$= [{_wR_c}^T \mid - {_wR_c}^T\, {_wt_c}]\, {_wX_i}$$
$$= [{_cR_w} \mid {_ct_w}]\, {_wX_i}$$
$$= [R|t]\, {_wX_i}$$



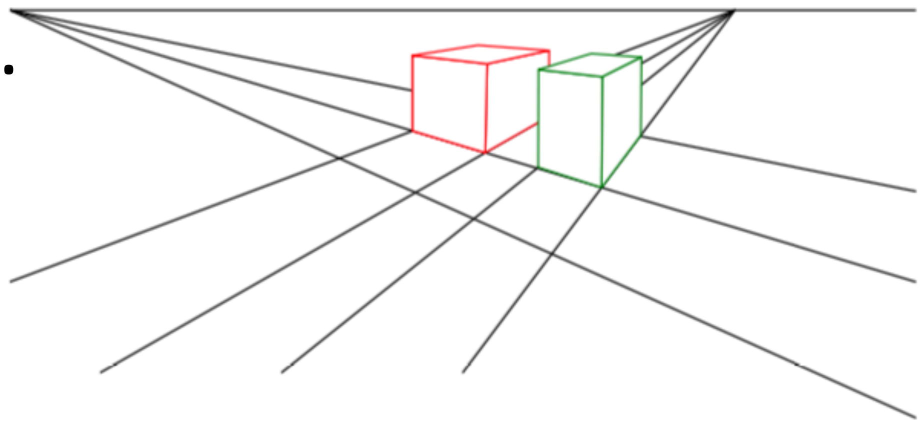
Camera Extrinsics: a Pose in 3D

- Conclusion: when people write ${}_cX_i = [R|t]\ {}_wX_i$ they are talking about (unintuitive) $[{}_cR_w \mid {}_ct_w]$
- We like use (intuitive) ${}_cX_i = {}_wR_c^T [I \mid -{}_wt_c]\ {}_wX_i$



Revision: Projective Camera Matrix

- Homogeneous coord.
- 3D TO 2D projection:



Camera-centric: $x = K[{}_cR_w \mid {}_ct_w]\, X = PX$

World-centric: $x = K\, {}_wR_c^T\, [I \mid -\, {}_wt_c]\, X = PX$

P = **same** 3x4 camera matrix

and K the 3x3 calibration $\boldsymbol{K} = \begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$

Looking at the (opaque) camera matrix

Can you interpret the columns of P with entities in the scene?

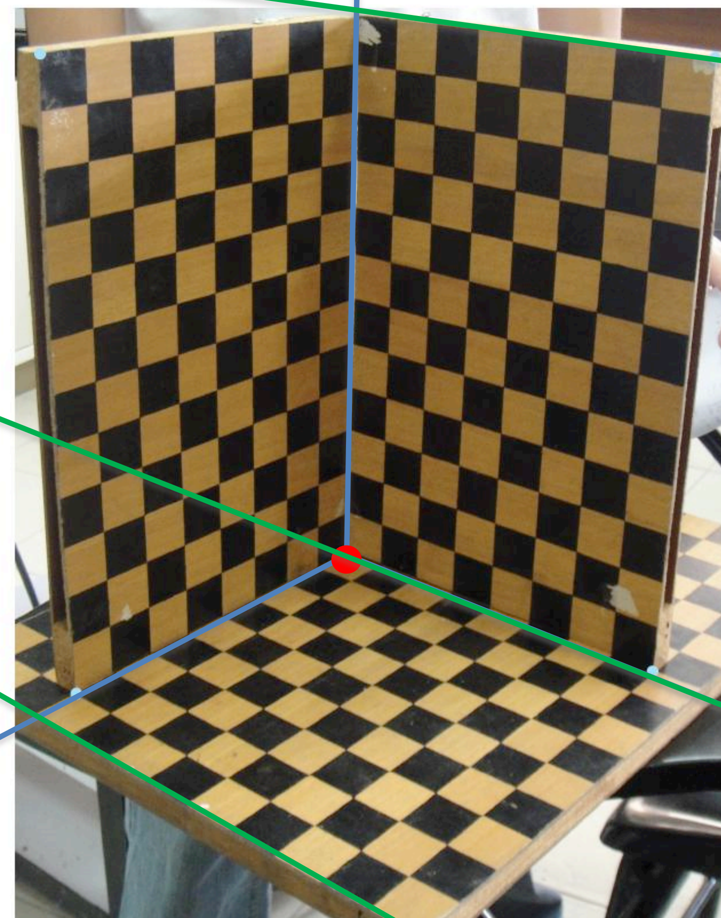
$$P = \begin{bmatrix} P^1 & P^2 & P^3 & P^4 \end{bmatrix}$$

Answer:
P^1 == the image of [1 0 0 0]
P^2 == the image of [0 1 0 0]
P^3 == the image of [0 0 1 0]
P^4 == the image of [0 0 0 1]

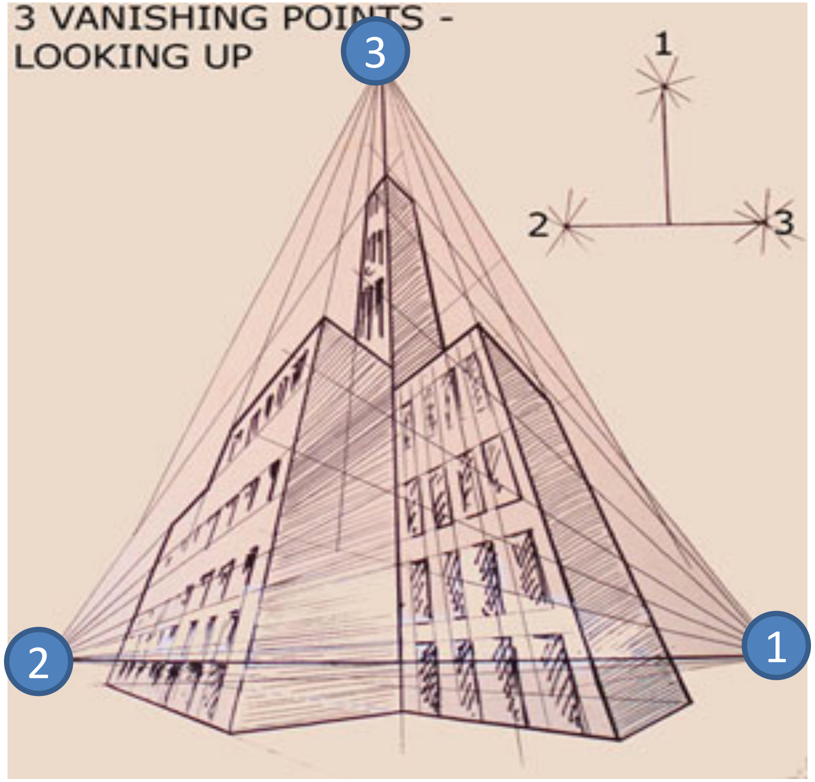
What are those ?
[0 0 0 1] is easy…

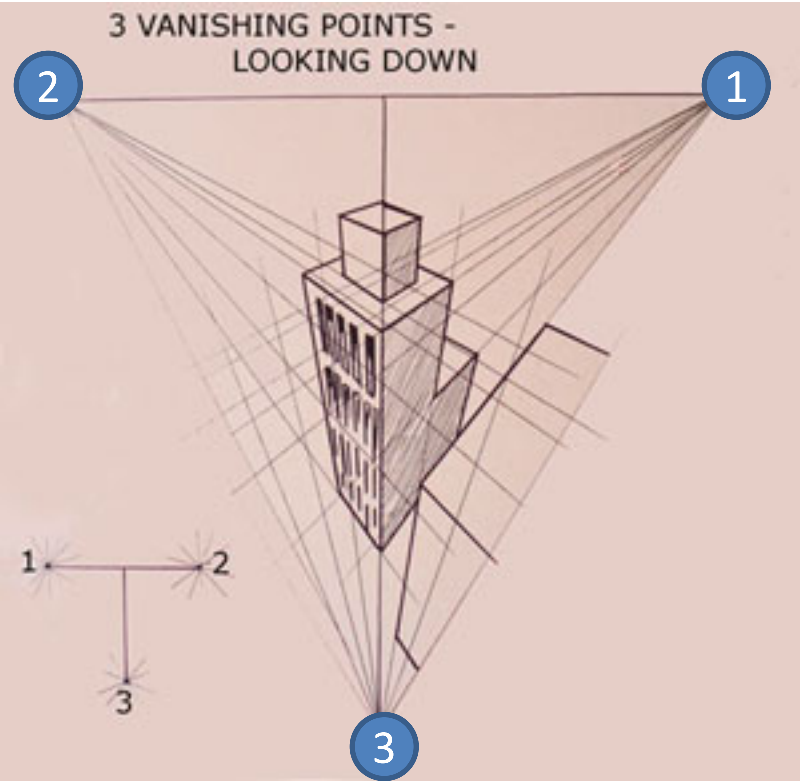
Answer:
[0 0 0 1] is the origin, so P4 is the image of the origin.

[1 0 0 0] is a point at infinity in the X-direction, so it is the vanishing point of all lines parallel with the X direction!



Vanishing points, revisited





Columns of P !

$$P = \begin{bmatrix} P^1 & P^2 & P^3 & P^4 \end{bmatrix}$$

P^4 is arbitrary: wherever you defined the world origin.

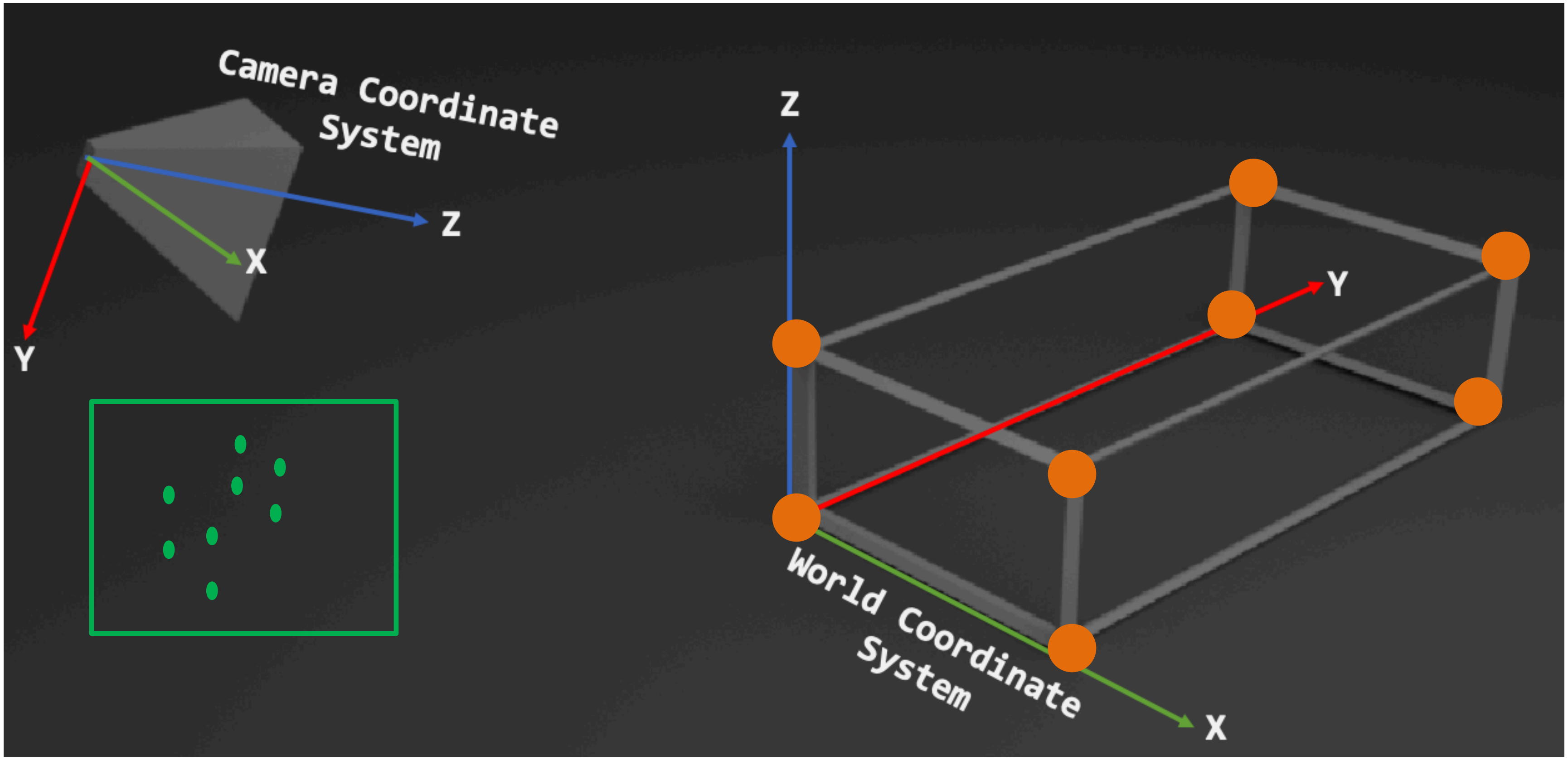
https://www.artinstructionblog.com/perspective-drawing-tutorial-for-artists-part-2

Back to Pose Estimation!

- Simple algorithm: just measure the coordinates of the origin and the three vanishing points?

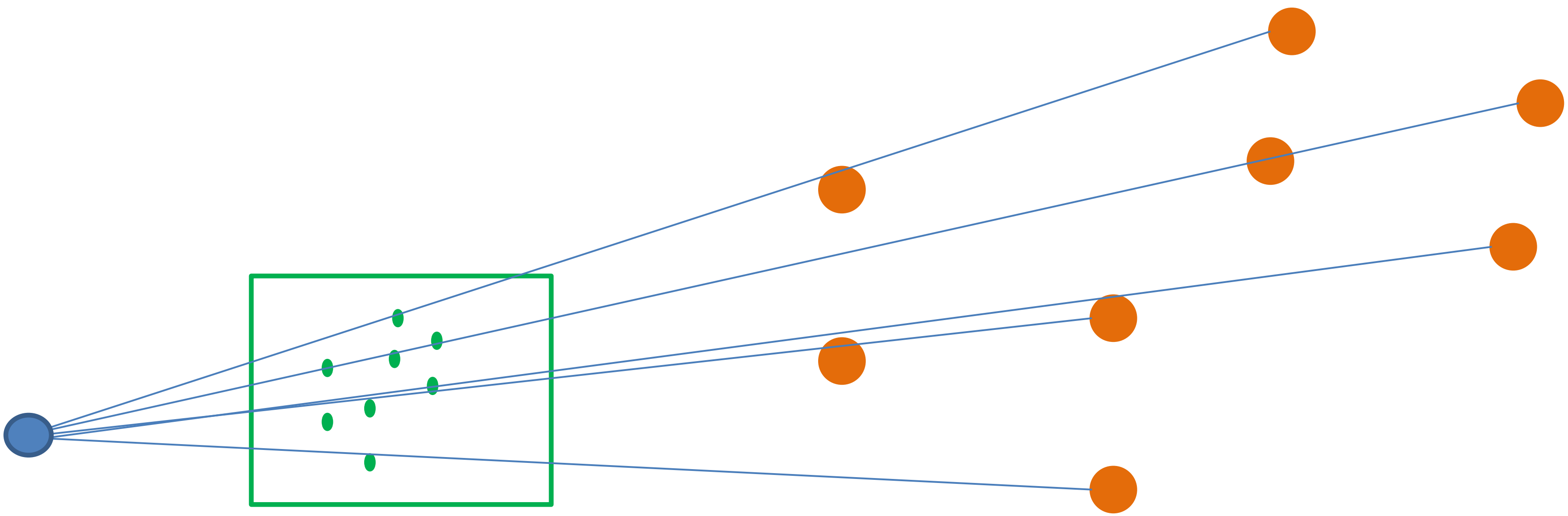
- Does not work ☹:
  - Columns are only measured up to a scale.
  - 4 points * 2DOF = only 8 DOF! Missing 11-8=3
  - 3 missing numbers are exactly those scales.

Least Squares Pose Estimation…



- Input:
  - A set of 2D measurements x_i of known 3D points X_i
  - Parametric model is camera matrix P, i.e., $x = f(X; P)$
- Output:
  - Best camera matrix P

Pose estimation = "Resectioning"



$$\mathbf{x} = f(\mathbf{X}_w; \mathbf{P}) = \mathbf{P}\mathbf{X}_w = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \cong \begin{bmatrix} s \cdot u \\ s \cdot v \\ s \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

$$\arg\min_{\hat{\mathbf{P}}} \sum_{i=1}^{N} ||\hat{\mathbf{P}}\mathbf{X}_w^i - \mathbf{x}^i||_2.$$

- Opposite of triangulation.

Pose estimation

$$\arg\min_{\hat{\mathbf{P}}} \sum_{i=1}^{N} ||\hat{\mathbf{P}}\mathbf{X}_w^i - \mathbf{x}^i||_2.$$

- In project 4, you will use `scipy.optimize.least_squares` to do exactly that. Working knowledge of 3D poses will be required.
- Note before we compute the 2D reprojection error we need to convert back *PX* to non-homogeneous coordinates:

$$x_i = \frac{p_{00}X_i + p_{01}Y_i + p_{02}Z_i + p_{03}}{p_{20}X_i + p_{21}Y_i + p_{22}Z_i + p_{23}}$$

$$y_i = \frac{p_{10}X_i + p_{11}Y_i + p_{12}Z_i + p_{13}}{p_{20}X_i + p_{21}Y_i + p_{22}Z_i + p_{23}}$$

https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.least_squares.html